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Correctness of high-level transformation systems relative to nested conditions

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We introduce the notions of nested constraints and application conditions, short nested conditions. For a category associated with a graphical representation such as graphs, conditions are a graphical and intuitive, yet precise formalism, well-suited to describe structural properties. We show that nested graph conditions are expressively equivalent to first-order graph formulas. A part of the proof includes transformations between two satisfiability notions of conditions, namely $\mathcal{M}$-satisfiability and $\mathcal{A}$-satisfiability. We consider a number of transformations on conditions that can be composed to construct constraint-guaranteeing and constraint-preserving application conditions, weakest preconditions, and strongest postconditions. The restriction of rule applications by conditions can be used to correct transformation systems by pruning transitions leading to states violating given constraints. Weakest preconditions and strongest postconditions can be used to verify the correctness of transformation systems with respect to pre- and postconditions.

Keywords: nested conditions, first-order graph formulas, high-level transformation systems, weak adhesive HLR categories, weakest preconditions, correctness.

Contents

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1. Introduction

In the context of increasingly larger and more complex systems that hardware and software engineers have to construct, visual modeling techniques can be expected to play a key role in the future. More than 30 years ago, graph replacement was proposed as a generalization of string replacement with the intent to yield an operational notation that captures and inherits the advantage of graphs – the ability to directly visualize relationships between elements. Especially the operational behavior of structure changing systems such as "mobile" systems (in the sense of dynamically changing communication topologies) is suited to be modeled by graph transformation rules (Ehrig et al. 1991).

Yet surprisingly, little effort was made to transfer the idea of a graph-based formalism to logical languages – until now. In this paper, we investigate nested conditions for graphs and high-level structures (Heckel and Wagner 1995; Koch et al. 2005; Ehrig et al. 2006b). For the category of graphs, nested conditions are a visual and intuitive, yet precise formalism, well-suited to describe structural properties of system states. We propose to model structure-changing systems by a graph or high-level replacement system and to model the required system properties by nested conditions.

However, the use of visual modeling techniques alone does not guarantee the correctness of a design. As standards for trustworthy systems grow, there is an increasing need for methods that support the development of correct transformation systems. We investigate two approaches: First, the construction of constraint-guaranteeing and weaker constraint-preserving conditions that restrict the applicability of rules to correct situations. Second, the verification of transformation systems with respect to pre- and postconditions. A well-known method is to construct a weakest precondition relative to the postcondition and to prove that the precondition implies the weakest precondition (Dijkstra 1976; Dijkstra and Scholten 1989).

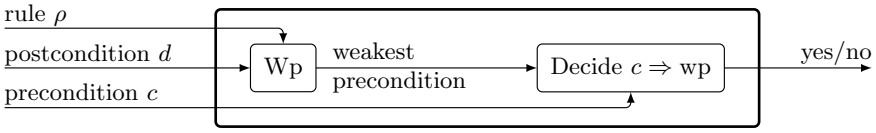


Figure 1. Decider for the correctness problem for transformation rules

As proposed in (Rensink 2004), we extend the existing notion of constraints and application conditions, conditions for short, to nested conditions and show that nested graph conditions and first-order graph formulas are expressively equivalent. We describe the construction of constraint-guaranteeing and constraint-preserving conditions as well as the computation of weakest preconditions and strongest postconditions with the help of certain basic transformations on conditions. Additionally, we investigate two notions of condition satisfiability and rule matching and show that the satisfiability and matching notions, respectively, are equivalent under reasonable assumptions. We are interested in conditions over graph-like structures. To avoid similar investigations for comparable structures, we abstract from specific definitions and conduct our research in the frame-

work of weak adhesive high-level replacement categories, HLR for short. Therefore, our results hold for replacement-capable structures such as Petri-nets, graphs, and hypergraphs. To illustrate the presented constructions, examples are given in the category of directed, labeled graphs.

The paper is organized as follows. In Section 2, we recall the notions of graphs and weak adhesive HLR categories as a framework for high-level structures and recount their basic properties. In Section 3, we introduce nested conditions for weak adhesive HLR categories. In Section 4 and Section 5, we investigate $\mathcal{M}$- and $\mathcal{A}$-satisfiability for conditions and $\mathcal{A}$- and $\mathcal{M}$-matching for rules. For weak adhesive HLR categories with epi-$\mathcal{M}$-factorizations, $\mathcal{M}$-initial object and a monomorphism class $\mathcal{M}$ strictly closed under decompositions, we show that $\mathcal{M}$- and $\mathcal{A}$-satisfiability, and $\mathcal{A}$- and $\mathcal{M}$-matching are expressively equivalent. In Section 6, we prove the basic transformation results, which are applied in Section 7 to yield constraint-guaranteeing and constraint-preserving application conditions as well as weakest preconditions and strongest postconditions. Finally, in Section 8, we investigate the expressiveness of nested graph conditions and show that nested graph conditions and first-order graph formulas are expressively equivalent. A conclusion including further work is given in Section 9.

This paper succeeds (Ehrig et al. 2006b), is a revised version of (Habel and Pennemann 2005; Habel and Pennemann 2006) and contains parts of (Habel et al. 2006) concerning transformation rules. Additionally, we present transformations between graph conditions and semantically equivalent first-order graph formulas, consider the example of a handover-protocol in cellular networks, and investigate the construction of strongest postconditions.

2. Graphs and high-level structures

In this paper, we consider graphs as a typical structure suited to model discrete aspects of system states. Our motivation for considering graphs is that they are a general structure and their graphical representation is able to directly visualize relationships between elements. In this section, we recall the basic notions of directed, labeled graphs (Ehrig 1979; Corradini et al. 1997) and generalize them to high-level structures (Ehrig et al. 1991). The idea behind the consideration of high-level structures is to avoid similar investigations for similar structures such as Petri-nets, graphs, and hypergraphs.

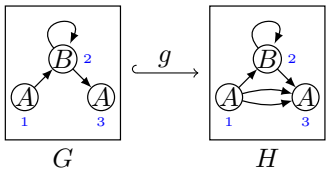
Directed, labeled graphs and graph morphisms are defined as follows.

Definition 1 (graphs and graph morphisms). Let $\mathrm{C} = \langle \mathrm{C_V}, \mathrm{C_E} \rangle$ be a fixed, finite, disjoint label alphabet. A *graph* over C is a system $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G, \mathrm{l}_G, \mathrm{m}_G)$ consisting of two finite sets V_G and E_G of *nodes* (or *vertices*) and *edges*, *source* and *target functions* $\mathrm{s}_G, \mathrm{t}_G \colon \mathrm{E}_G \to \mathrm{V}_G$, and two *labeling functions* $\mathrm{l}_G \colon \mathrm{V}_G \to \mathrm{C_V}$ and $\mathrm{m}_G \colon \mathrm{E}_G \to \mathrm{C_E}$. A graph with empty set of nodes is *empty* and denoted by $\emptyset$. A *graph morphism* $g \colon G \to H$ consists of two functions $g_\mathrm{V} \colon \mathrm{V}_G \to \mathrm{V}_H$ and $g_\mathrm{E} \colon \mathrm{E}_G \to \mathrm{E}_H$ that preserve sources, targets, and labels, that is, $\mathrm{s}_H \circ g_\mathrm{E} = g_\mathrm{V} \circ \mathrm{s}_G$, $\mathrm{t}_H \circ g_\mathrm{E} = g_\mathrm{V} \circ \mathrm{t}_G$, $\mathrm{l}_H \circ g_\mathrm{V} = \mathrm{l}_G$, and $\mathrm{m}_H \circ g_\mathrm{E} = \mathrm{m}_G$. A morphism g is *injective* (*surjective*) if g_V and g_E are injective (surjective), and an

isomorphism if it is both injective and surjective. The *composition* $h \circ g$ of g with a morphism $h\colon H \to M$ consists of the composed functions $h_{\mathrm{V}} \circ g_{\mathrm{V}}$ and $h_{\mathrm{E}} \circ g_{\mathrm{E}}$.

For simplicity, we sometimes consider a single labeling function $\mathrm{l}_G\colon \mathrm{V}_G + \mathrm{E}_G \to \mathrm{C}_{\mathrm{V}} \cup \mathrm{C}_{\mathrm{E}}$ with $\mathrm{l}_G(\mathrm{V}_G) \subseteq \mathrm{C}_{\mathrm{V}}$ and $\mathrm{l}_G(\mathrm{E}_G) \subseteq \mathrm{C}_{\mathrm{E}}$, where $+$ denotes the disjoint union. Of course, two labeling functions for nodes and edges, respectively, can be composed into one, and vice versa.

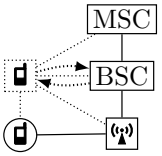
Example 1. Consider the graphs G and H and the injective graph morphism $g\colon G \to H$ given in the figure below. G and H are graphs over the label alphabet $\mathrm{C} = \langle \{A, B\}, \{\square\} \rangle$, where the symbol $\square$ stands for the invisible edge label and is not drawn. Note that the graph G contains a loop; the graph H contains two parallel edges and a loop.



In drawings of graphs, nodes are drawn by circles carrying the node label inside; edges are drawn by arrows pointing from the source to the target node and the edge label is placed next to arrow. Arbitrary graph morphisms are drawn by usual arrows "$\to$"; the use of "$\hookrightarrow$" indicates an injective graph morphism. For graph morphisms, the mapping of nodes is depicted by indices, if necessary.

Example 2 (GSM graph). In the Global System for Mobile Communications (GSM), mobile stations (MS) are connected by wireless links with the cellular network of their operators. Such a network consists of mobile services switching centers (MSC) managing a number of base station controllers (BSC) which control a number of base transceiver stations (BTS). In this running example, we model selected aspects of the handover procedure (ETSI TS 100 527) between two base station controllers of a mobile services switching center, the so-called inter-BSC-intra-MSC handover. States of the GSM are represented as graphs. The following elements are modeled as labeled nodes: MSC and BSC, base transceiver stations, mobile stations, and mobile station records, i.e. mobile stations as seen by the network and its components.

Relations between these elements are modeled by labeled edges: physical connections including wireless ones are modeled by solid edges (—), logical or virtual connections by dotted edges (·····), while certain messages exchanged between mobile stations and BSCs in the stages of the handover procedure are modeled by request ($\overset{req}{\cdots\blacktriangleright}$), command ($\overset{cmd}{\cdots\blacktriangleright}$) and complete edges ($\overset{cpl}{\cdots\blacktriangleright}$), respectively. Undirected edges represent a pair of directed edges in opposite directions.



In the sense of category theory, graphs and graph morphisms form a category.

Fact 1 (Ehrig 79). Graphs and graph morphisms form the category **Graphs**.

The following considerations are done on the level of objects and morphisms with specific properties, so-called weak adhesive HLR categories. The objects can be all kinds of structures which are of interest in computer science and mathematics, e.g. Petri-nets, (hyper)graphs, and algebraic specifications, the morphisms can be net, (hyper)graph, and specification morphisms, respectively. Readers interested in the category-theoretic background of these concepts may consult, e.g. (Adamek et al. 1990; Ehrig et al. 2006c).

For a category $\mathcal{C}$, let $\mathcal{A}$ be the class of all morphisms.

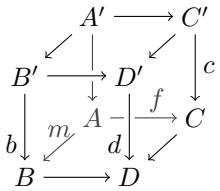
Definition 2 (weak adhesive HLR category). A category $\mathcal{C}$ with a morphism class $\mathcal{M} \subseteq \mathcal{A}$ is a *weak adhesive HLR category*, if the following properties hold:

(1) $\mathcal{M}$ is a class of monomorphisms closed under isomorphisms, composition, and decomposition. I.e. for morphisms $g \circ f$: $f \in \mathcal{M}$, g isomorphism (or vice versa) implies $g \circ f \in \mathcal{M}$; $f, g \in \mathcal{M}$ implies $g \circ f \in \mathcal{M}$; and $g \circ f \in \mathcal{M}$, $g \in \mathcal{M}$ implies $f \in \mathcal{M}$.

(2) $\mathcal{C}$ has pushouts and pullbacks along $\mathcal{M}$-morphisms, i.e. pushouts and pullbacks, where at least one of the given morphisms is in $\mathcal{M}$, and $\mathcal{M}$-morphisms are closed under pushouts and pullbacks, i.e. given a pushout (1), $m \in \mathcal{M}$ implies $n \in \mathcal{M}$ and, given a pullback (1), $n \in \mathcal{M}$ implies $m \in \mathcal{M}$.

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle m}\downarrow & (1) & \downarrow{\scriptstyle n} \\ B & \longrightarrow & D \end{array}$$

(3) Pushouts in $\mathcal{C}$ along $\mathcal{M}$-morphisms are weak VK-squares, i.e. for any commutative cube in $\mathcal{C}$ where we have the pushout with $m \in \mathcal{M}$ and ($f \in \mathcal{M}$ or $b, c, d \in \mathcal{M}$) in the bottom and the back faces are pullbacks, it holds: the top is pushout iff the front faces are pullbacks.



Fact 2 (Ehrig et al. 2006c). The category $\langle \textbf{Graphs}, Inj \rangle$ of graphs with class Inj of all injective graph morphisms is a weak adhesive HLR category. Further examples of weak adhesive HLR categories are the categories of hypergraphs with class of all injective hypergraph morphisms, place-transition nets with class of all injective net morphisms, and algebraic specifications with class of all strict injective specification morphisms.

Weak adhesive HLR categories have a number of nice properties, called HLR properties (Ehrig et al. 1991).

Fact 3 (HLR properties of weak adhesive HLR categories). Given a weak adhesive HLR category $\langle \mathcal{C}, \mathcal{M} \rangle$, the following HLR conditions are satisfied.

(1) Pushouts along $\mathcal{M}$-morphisms are pullbacks.

(2) Pushout-pullback decomposition. If the diagram (1)+(2) is a pushout, (2) a pullback, $w \in \mathcal{M}$ and ($l \in \mathcal{M}$ or $c \in \mathcal{M}$), then (1) and (2) are pushouts and also pullbacks.

$$\begin{array}{ccccc} A & \xrightarrow{c} & C & \xrightarrow{r} & E \\ {\scriptstyle l}\downarrow & (1) & {\scriptstyle s}\downarrow & (2) & \downarrow{\scriptstyle v} \\ B & \xrightarrow[u]{} & D & \xrightarrow[w]{} & F \end{array}$$

(3) Uniqueness of pushout complements for $\mathcal{M}$-morphisms. Given morphisms $c\colon A \to C$ in $\mathcal{M}$ and $s\colon C \to D$, then there is up to isomorphism at most one B with $l\colon A \to B$ and $u\colon B \to D$ such that diagram (1) is a pushout.

Proof. See (Lack and Sobociński 2004; Ehrig et al. 2006c). □

For some results in this paper, we use the following properties.

Definition 3. Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category. $\langle \mathcal{C}, \mathcal{M} \rangle$ has *pushouts*, if for each pair of arbitrary morphisms, there is a pushout. $\langle \mathcal{C}, \mathcal{M} \rangle$ has an *epi-$\mathcal{M}$-factorization* if, for every morphism, there is an epi-mono-factorization with monomorphism in $\mathcal{M}$ and this decomposition is unique up to isomorphism. $\langle \mathcal{C}, \mathcal{M} \rangle$ has an *$\mathcal{M}$-initial object* if there is some object I in $\mathcal{C}$ such that, for every object G in $\mathcal{C}$, there exists a unique morphism $i_G\colon I \to G$ and i_G is in $\mathcal{M}$. We refer to i_G as *initial $\mathcal{M}$-morphism* to G. $\mathcal{M}$ is *strictly* closed under decomposition if $g \circ f \in \mathcal{M}$ implies $f \in \mathcal{M}$.

Fact 4. The weak adhesive HLR category $\langle \mathbf{Graphs}, \mathit{Inj} \rangle$ with *Inj* the class of all injective graph morphisms has pushouts, an epi-*Inj*-factorization, an *Inj*-initial object, and *Inj* is strictly closed under decompositions.

Proof. The category **Graphs** has pushouts, see e.g. (Ehrig et al. 2006c), Fact A.19. It has epi-mono factorizations (A.15) and the monomorphisms are exactly those morphisms that are injective (A.14 and Fact 2.15). The empty graph is an *Inj*-initial object in $\langle \mathbf{Graphs}, \mathit{Inj} \rangle$. The class *Inj* of injective graph morphisms is strictly closed under decompositions because the class of injective mappings has this property. □

3. Conditions and rules

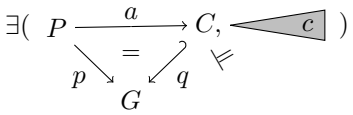
In search for a graphical formalism to specify sets of objects as well as morphisms, we investigate nested conditions for graphs and high-level structures. We use the framework of weak adhesive HLR categories and introduce nested conditions and rules for high-level structures like Petri nets, (hyper)graphs, and algebraic specifications.

Assumption 1. We assume that $\langle \mathcal{C}, \mathcal{M} \rangle$ is a weak adhesive HLR category with $\mathcal{M}$-initial object I.

Syntactically, nested conditions may be seen as a tree of morphisms equipped with certain logical symbols such as quantifiers and connectives.

Definition 4 (conditions). A *(nested) condition* over an object P is of the form true or $\exists(a, c)$, where $a\colon P \to C$ is a morphism and c is a condition over C. Moreover, Boolean formulas over conditions over P yield conditions over P, i.e., $\neg c$ and $\wedge_{j\in J}\, c_j$ are (Boolean) conditions over P, where J is an index set and c, $(c_j)_{j\in J}$ are conditions over P. Additionally, $\exists a$ abbreviates $\exists(a, \text{true})$, $\forall(a, c)$ abbreviates $\neg\exists(a, \neg c)$, false abbreviates $\neg$true, $\vee_{j\in J}\, c_j$ abbreviates $\neg\wedge_{j\in J}\, \neg c_j$ and $c \Rightarrow d$ abbreviates $\neg c \vee d$.

Every object and morphism *satisfies* true. A morphism p *satisfies* a condition $\exists(a, c)$, if there exists a morphism q in $\mathcal{M}$ such that $q \circ a = p$ and q satisfies c.



An object G *satisfies* a condition $\exists(a, c)$, if the condition is over the initial object I and the initial morphism $i_G\colon I \to G$ satisfies the condition. The satisfaction of conditions by objects and morphisms is extended onto Boolean conditions in the usual way. We write $G \models c$ resp. $p \models c$ to denote that the object G resp. the morphism p satisfies c. Two conditions c and c' are *equivalent*, denoted by $c \equiv c'$, if, for all morphisms p, $p \models c$ iff $p \models c'$.

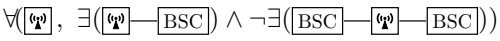
In the context of objects, conditions are also called *constraints* (Heckel and Wagner 1995) and, in the context of rules, conditions are also called *application conditions*.

Example 3. For graph morphisms with domain $\bigcirc_1$, the graph condition $c = \exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc) \vee \exists(\bigcirc_1 \to \bigcirc_1 \leftarrow \bigcirc)$ has the meaning "The image of the node has a proper outgoing or incoming edge". For graphs, one may consider the universal closure $\forall(\emptyset \to \bigcirc_1,\ c)$ with the meaning "All nodes have an outgoing or incoming proper edge".

Notation. For a morphism $a\colon P \to C$ in a condition, we just depict the codomain C, if the domain P can be unambiguously inferred, i.e. if it is known over which object a condition is. For instance, as constraints are always over the initial object, the constraint $\forall(\emptyset \to \bigcirc_1,\ \exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc_2))$ meaning "Every node has an outgoing edge to a distinct node" can be represented by $\forall(\bigcirc_1,\ \exists(\bigcirc_1 \to \bigcirc_2))$.

Example 4 (GSM conditions). Consider the GSM graph model in Example 2. Structural requirements on the GSM states can be described by graph constraints, i.e. graph conditions over the empty graph.

(c_1) Every transceiver station has exactly one controller:

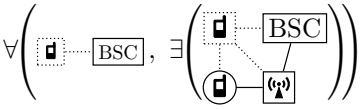


(c_2) No mobile station is associated to two or more mobile station records and vice versa:

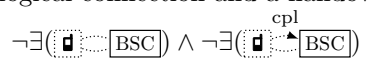
(c_3) Every mobile station record is associated with a base transceiver station:

(c_4) There is no mobile station connected to two or more base transceiver stations and no mobile station with two channels at a transceiver station:

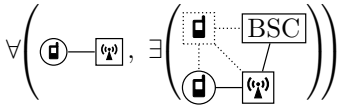
(c_5) Every logical connection between a base transceiver station and a mobile station record correspond with a physical connection between the network and the mobile station:



(c_6) Between a base transceiver station and mobile station record, there are not two logical connections, nor a logical connection and a handover complete message edge:



(c_7) Every wireless connection between a mobile and a transceiver station is reflected by the network:



Let $\mathit{consistency} = \wedge_{j=1}^{7} c_j$ denote the conjunction of the GSM conditions. A GSM graph is said to be *consistent*, if it satisfies the graph condition *consistency*.

Remark. Nested conditions, as proposed in (Rensink 2004), subsume the previous notions of basic constraints and application conditions. For every condition in the sense of (Koch et al. 2005; Ehrig et al. 2006b), there is an equivalent condition according to Definition 4. The presented notion of conditions correspond to the notions in (Habel and Pennemann 2005) with one difference: Only constraints over the initial object are considered. This is no restriction, as every constraint over P in the sense of (Habel and Pennemann 2005) can be transformed into an equivalent constraint over the initial object I, e.g. $\exists(a, c)$ may be transformed into $\forall(i_P, \exists(a, c))$. By this modification, we obtain a common notion of equivalence for objects and morphisms.

In the following, we show that the concepts of constraints and application conditions coincide in terms of equivalence. We use this fact in Section 4 by defining transformations for conditions that may be used for constraints as well as application conditions. Note that the add-on [in $\mathcal{M}$] is optional.

Fact 5 (equivalence). For conditions c, c' over I, for all morphisms p [in $\mathcal{M}$], $p \models c \Leftrightarrow p \models c'$ iff for all objects G, $G \models c \Leftrightarrow G \models c'$.

Proof. **Only if.** Assume, for all morphisms p [in $\mathcal{M}$], $p \models c \Leftrightarrow p \models c'$. In particular, this means for objects G, $i_G \models c \Leftrightarrow i_G \models c'$ where i_G is the initial $\mathcal{M}$-morphisms to G. By Definition 4, we conclude for every object G, $G \models c \Leftrightarrow G \models c'$. **If.** Conversely, if for every object G, $G \models c \Leftrightarrow G \models c'$, then $i_G \models c \Leftrightarrow i_G \models c'$ where i_G is the initial $\mathcal{M}$-morphism to G. Let $p\colon I \to G$ be any morphism. By the $\mathcal{M}$-initiality of I, we know that $p = i_G$ [in $\mathcal{M}$], therefore we conclude for all morphisms p [in $\mathcal{M}$], $p \models c \Leftrightarrow p \models c'$. □

Conditions are allowed to consist of morphisms not in $\mathcal{M}$. In context of rules, such conditions are useful as application conditions to specify the identification/non-identification of elements. However, for constraints, the use of morphisms not in $\mathcal{M}$ does not increase the expressiveness. We show that every condition over I can be transformed into $\mathcal{M}$-normal form.

Definition 5 ($\mathcal{M}$-normal form). A condition is in *$\mathcal{M}$-normal form*, if for all subconditions of the form $\exists(a, c)$, the morphism a is in $\mathcal{M}$.

Fact 6 ($\mathcal{M}$-normal form). For every condition c over I, there is an equivalent condition $\mathcal{M}\mathrm{NF}(c)$ in $\mathcal{M}$-normal form.

Construction. Define $\mathcal{M}\mathrm{NF}(\mathrm{true}) = \mathrm{true}$ and

$$\mathcal{M}\mathrm{NF}(\exists(a, c')) = \begin{cases} \mathrm{false} & \text{if } a \notin \mathcal{M} \\ \exists(a,\ \mathcal{M}\mathrm{NF}(c')) & \text{otherwise.} \end{cases}$$

For Boolean conditions, the transformation is extended in the usual way, i.e. $\mathcal{M}\mathrm{NF}(\neg c') = \neg\mathcal{M}\mathrm{NF}(c')$ and $\mathcal{M}\mathrm{NF}(\wedge_{j\in J}\ c_j) = \wedge_{j\in J}\ \mathcal{M}\mathrm{NF}(c_j)$.

Proof. Obviously, for every condition c, $\mathcal{M}\mathrm{NF}(c)$ is in $\mathcal{M}$-normal form. By structural induction, we show for every $p \in \mathcal{M}$, $p \models c$ iff $p \models \mathcal{M}\mathrm{NF}(c)$. **Basis.** For $c = \mathrm{true}$, we have $c = \mathrm{true} = \mathcal{M}\mathrm{NF}(\mathrm{true}) = \mathcal{M}\mathrm{NF}(c)$. **Hypothesis.** Assume, for every $p \in \mathcal{M}$, $p \models c'$ iff $p \models \mathcal{M}\mathrm{NF}(c')$ and $p \models c_j$ iff $p \models \mathcal{M}\mathrm{NF}(c_j)$ for every $j \in J$. **Step.** Let $c = \exists(a, c')$. If $a \in \mathcal{M}$, then we have, for every $p \in \mathcal{M}$, $p \models \mathcal{M}\mathrm{NF}(c) = \mathcal{M}\mathrm{NF}(\exists(a, c')) = \exists(a,\ \mathcal{M}\mathrm{NF}(c'))$ iff $p \models \exists(a, c') = c$ by the definition of $\mathcal{M}\mathrm{NF}$ and the induction hypothesis. Otherwise $a \notin \mathcal{M}$, and $\mathcal{M}\mathrm{NF}(\exists(a, c')) = \mathrm{false}$. We show, for every $p \in \mathcal{M}$, $p \models \exists(a, c')$ iff $p \models \mathrm{false}$. **Only if.** Assume morphism p in $\mathcal{M}$ satisfies $\exists(a, c')$. Then there is a morphism q in $\mathcal{M}$ with $p = q \circ a$. However, p, q in $\mathcal{M}$ and $\mathcal{M}$ closed under decomposition implies $a \in \mathcal{M}$, contradiction. **If.** No morphism satisfies false, therefore for every morphism p, $p \models \mathrm{false}$ implies $p \models \exists(a, c')$. For Boolean formulas over conditions, the statement follows directly from the definitions and the inductive hypothesis. For $c = \neg c'$, we have, for every $p \in \mathcal{M}$, $p \models \mathcal{M}\mathrm{NF}(c) = \mathcal{M}\mathrm{NF}(\neg c') = \neg\mathcal{M}\mathrm{NF}(c')$ iff $p \models \neg c' = c$. For $c = \wedge_{j\in J}\ c_j$, we have, for every $p \in \mathcal{M}$, $p \models \mathcal{M}\mathrm{NF}(c) = \mathcal{M}\mathrm{NF}(\wedge_{j\in J}\ c_j) = \wedge_{j\in J}\ \mathcal{M}\mathrm{NF}(c_j)$ iff $p \models \wedge_{j\in J}\ c_j = c$. □

Remark ($\mathcal{M}$-satisfiability). The satisfaction of a condition is established by the presence and absence of certain morphisms from the objects within the condition to the tested object. The presented satisfiability notion restricts these morphisms to the class $\mathcal{M}$: no identification of nodes and edges is allowed. Hence explicit counting such as the existence/non-existence of n nodes or n edges is easily expressible. We speak of *$\mathcal{M}$-satisfiability* and *$\mathcal{M}$-satisfiable conditions*.

Beside $\mathcal{M}$-satisfiability one may consider *$\mathcal{A}$-satisfiability*/*$\mathcal{A}$-satisfiable conditions*, where $\mathcal{A}$ denotes the class of all morphisms. The definition of $\mathcal{A}$-satisfiability is obtained from the one of $\mathcal{M}$-satisfiability, by replacing all occurrences of $\mathcal{M}$ by $\mathcal{A}$, or by deleting all occurrences of "in $\mathcal{M}$", respectively.

Definition 6 ($\mathcal{A}$-satisfiability). Every object and morphism *$\mathcal{A}$-satisfies* true. An object G *$\mathcal{A}$-satisfies* a condition $\exists(a, c)$, if the condition is over the initial object I and the initial morphism $i_G\colon I \to G$ $\mathcal{A}$-satisfies the condition. A morphism p *$\mathcal{A}$-satisfies* a condition $\exists(a, c)$, if there exists a morphism q such that $q \circ a = p$ and q $\mathcal{A}$-satisfies c. The $\mathcal{A}$-satisfaction of conditions by objects and morphisms is extended onto Boolean conditions in the usual way. We write $G \models_{\mathcal{A}} c$ resp. $p \models_{\mathcal{A}} c$ to denote that the object G resp. the morphism p $\mathcal{A}$-satisfies c. Two conditions c and c' are *$\mathcal{A}$-equivalent*, denoted by $c \equiv_{\mathcal{A}} c'$, if, for all morphisms p, $p \models_{\mathcal{A}} c$ iff $p \models_{\mathcal{A}} c'$.

Remark ($\mathcal{A}$-satisfiability). $\mathcal{A}$-satisfiability allows nodes or edges of the conditions to be identified and is closely related to the satisfiability of first-order formulas as indicated in Section 8 for the case of directed, labeled graphs. In Section 4, we will see that, under reasonable assumptions, $\mathcal{A}$- and $\mathcal{M}$-satisfiability are expressively equivalent. Unless explicitly stated, theorems concern $\mathcal{M}$-satisfiable conditions.

Example 5 ($\mathcal{A}$-satisfiability). The meaning of the graph condition $\exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc)$ for graph morphisms with domain $\bigcirc_1$ w.r.t. $\mathcal{A}$-satisfiability is: "There exists an outgoing edge (proper edge or loop)".

In the following, we consider rules with application conditions (Ehrig et al. 2006b; Habel and Pennemann 2005). Examples and pointers to the literature can be found in (Ehrig 1979; Corradini et al. 1997).

Definition 7 (rules). A *plain rule* $\rho = \langle L \hookleftarrow K \hookrightarrow R \rangle$ consists of two morphisms in $\mathcal{M}$ with a common domain K. L is called the left-hand side, R the right-hand side, and K the interface. An *application condition* $\mathrm{ac} = \langle \mathrm{ac}_L, \mathrm{ac}_R \rangle$ for ρ consists of two conditions over L and R, respectively. A *rule* $\hat{\rho} = \langle \rho, \mathrm{ac} \rangle$ consists of a plain rule ρ and an application condition ac for ρ.

$$\begin{array}{ccccc} \mathrm{ac}_L & & & & \mathrm{ac}_R \\ \Vdash & L & \xleftarrow{\ l\ } & K & \xrightarrow{\ r\ } & R & \Vdash \\ & m\downarrow & (1) & \downarrow & (2) & \downarrow m^* & \\ & G & \longleftarrow & D & \longrightarrow & H & \end{array}$$

Given a plain rule ρ and a morphism $K \to D$, a *direct derivation* consists of two pushouts (1) and (2). We write $G \Rightarrow_{\rho,m,m^*} H$ or short $G \Rightarrow_{\rho} H$and say that m is the *match* and m^* is the *comatch* of ρ in H. Given a rule $\hat{\rho} = \langle \rho, \mathrm{ac} \rangle$ and a morphism $K \to D$, there is a *direct derivation* $G \Rightarrow_{\hat{\rho},m,m^*} H$ if $G \Rightarrow_{\rho,m,m^*} H$, $m \models \mathrm{ac}_L$, and $m^* \models \mathrm{ac}_R$.

Example 6. Consider the graph replacement rule with inclusions given in Figure 2. Then there is an injective graph morphism from the left-hand side L of the rule to the graph G, fixed by the numbers beside the nodes, that satisfies the so-called gluing condition (Ehrig 1979). The removement of $m(L - K)$ yields the graph D and the addition $R - K$ yields the graph H. The match m satisfies the left application condition $\neg\exists($  $\to$  $)$; the comatch m^* satisfies the right application condition $\neg\exists(\bigcirc\!\to\!\bigcirc \to \bigcirc$  $)$. Both application conditions guarantee that multiple edges cannot arise.

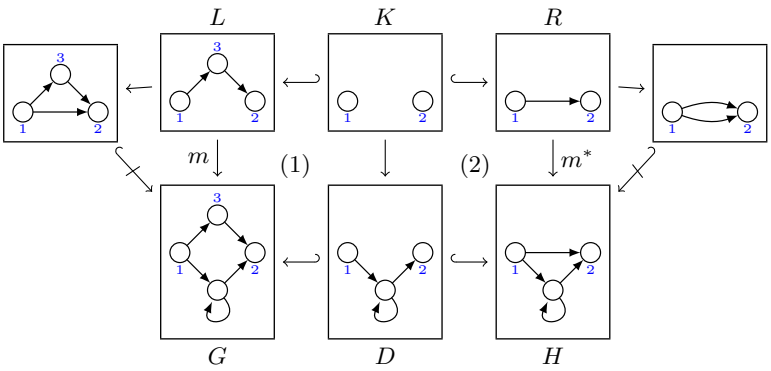


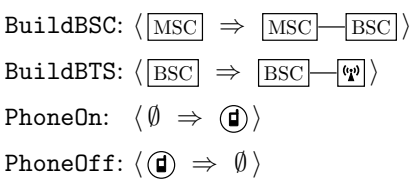
Figure 2. Application of a graph replacement rule

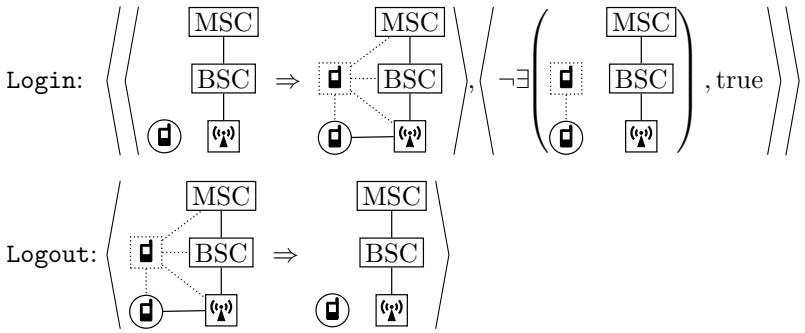
Remark ($\mathcal{A}$- and $\mathcal{M}$-matching). Matching, also called $\mathcal{A}$-*matching*, is the process of finding arbitrary matching morphisms. Besides $\mathcal{A}$-matching, one may consider only matches in $\mathcal{M}$, called $\mathcal{M}$-*matching* and denoted by $\Rightarrow_{\mathcal{M}}$. $\mathcal{M}$-matching restricts the applicability of rules: no identification of nodes and edges is allowed. Moreover, $\mathcal{M}$-matching allows explicit counting such as the deletion of n nodes or edges. In Section 5, it will be shown that, under reasonable assumptions, $\mathcal{A}$ and $\mathcal{M}$-matching are expressively equivalent.

A set of transformation rules constitutes a transformation system.

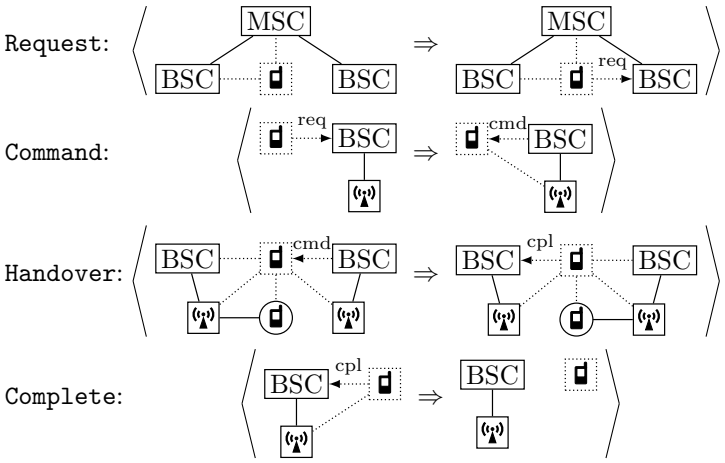
Definition 8 (transformation system). A *transformation system* is a set of rules $\mathcal{R}$. A *derivation* is a sequence of direct derivations of rules in $\mathcal{R}$.

Example 7 (GSM transformation system). Consider the GSM model in Example 2. For simplicity, we select $\mathcal{M}$-matching and denote a rule $\langle L \hookleftarrow K \hookrightarrow R \rangle$ shortly by $\langle L \Rightarrow R \rangle$, where K consists of all elements common to L and R. The dynamics of the GSM network is modeled by six graph transformation rules: `BuildBSC` and `BuildBTS` model the extension of infrastructure. `PhoneOn` and its inverse `PhoneOff` model the operational status of a mobile station. `Login` and its inverse `Logout` model the creation and deletion of a mobile station record.





The operational behavior of the handover protocol is modeled by the graph transformation rules `Request`, `Command`, `Handover` and `Complete`, representing a subdivision of the handover operation into four phases. In the first phase, the necessity of the handover is detected by the base station controller based on informations last received from the transceiver station. A handover `Request` message is sent to the MSC which selects a new BSC and forwards the message to it. In the second phase, the new BSC transfers the handover `Command` through the old BSC to the MS, assigning it to a radio channel of one of its base transceiver stations. In the third phase, the MS disconnects the old radio channel and establishes the radio channel with the new BSC. If successful, the connection to the old BSC is reestablished via the new BSC through its transceiver station. In the last phase, the MS sends the handover `Complete` message to the old BSC, which releases the old radio channels on its transceiver station.



The question arises, whether or not the GSM transformation system is correct with respect to the pre- and postcondition *consistency*, i.e. whether the application of the GSM rules on a consistent GSM graph always results in a consistent graph.

4. Satisfiability of conditions

In this section, we investigate the different satisfiability notions for conditions and show that $\mathcal{A}$-satisfiability and $\mathcal{M}$-satisfiability are expressively equivalent. First, there is a transformation from $\mathcal{A}$- to $\mathcal{M}$-satisfiability. In case pushouts for arbitrary pairs of morphisms exist, this transformation of conditions has some similarities to the transformation of constraints into application conditions in (Ehrig et al. 2006b) and the original paper of (Heckel and Wagner 1995). In the case that the existence of pushouts cannot be guaranteed, one can resort to a modified transformation which requires an $\mathcal{M}$-initial object and makes use of the existence of pushouts along $\mathcal{M}$-morphisms.

Theorem 1 (from $\mathcal{A}$- to $\mathcal{M}$-satisfiability). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object and epi-$\mathcal{M}$-factorization. There is a transformation Msat such that, for every condition c and for every morphism p, $p \models \mathrm{Msat}(c) \Leftrightarrow p \models_{\mathcal{A}} c$, and for every condition c over I and for object G, $G \models \mathrm{Msat}(c) \Leftrightarrow G \models_{\mathcal{A}} c$.

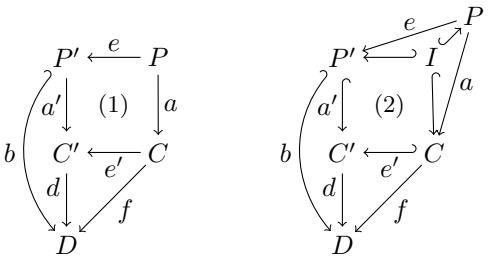
$\mathcal{A}$-satisfiable conditions allow elements to be identified. The idea of the following construction is to consider a disjunction of $\mathcal{M}$-satisfiable conditions, one for each possible level of identification. The construction of $\mathrm{Msat}(c)$ with single parameter of type condition is defined by the auxiliary transformation $\mathrm{Msat}(e, c)$ where the first parameter is of type epimorphism and the second parameter is of type condition.

Construction. For a condition c over P, define $\mathrm{Msat}(c) = \vee_{e \in \mathcal{E}'} \exists(e,\ \mathrm{Msat}(e, c))$ where the set $\mathcal{E}'$ ranges over all epimorphisms with domain P. For every epimorphism e, the transformation $\mathrm{Msat}(e, c)$ is defined inductively by $\mathrm{Msat}(e, \mathrm{true}) = \mathrm{true}$ and

$$\mathrm{Msat}(e, \exists(a, c')) = \vee_{d \in \mathcal{E}} \exists(b,\ \mathrm{Msat}(f, c'))$$

where, in the case that $\langle \mathcal{C}, \mathcal{M} \rangle$ has pushouts, (1) is the pushout of the morphisms a and e leading to morphisms $a' \colon P' \to C'$ and $e' \colon C \to C'$ and the set $\mathcal{E}$ ranges over all epimorphisms d with domain C' such that $b = d \circ a'$ is in $\mathcal{M}$ and $f = d \circ e'$.

Alternatively, construct the pushout (2) of the initial $\mathcal{M}$-morphisms $i_C \colon I \to C$ and $i_{P'} \colon I \to P'$ leading to morphisms $a' \colon P' \to C'$ and $e' \colon C \to C'$ and the set $\mathcal{E}$ ranges over all epimorphisms d with domain C' such that $b = d \circ a'$ is in $\mathcal{M}$, $f = d \circ e'$ and $b \circ e = f \circ a$.



For Boolean formulas over conditions, the transformations $\mathrm{Msat}(\_)$ and $\mathrm{Msat}(\_, \_)$ are extended in the usual way.

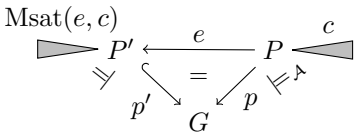
Example 8. The meaning of condition $c = \exists(\bigcirc_1\ \bigcirc_2 \to \bigcirc_1 \to \bigcirc_2)$ for graph morphisms w.r.t. $\mathcal{A}$-satisfiability is "There exists an edge from the image of node 1 to the image of node 2". $\mathrm{Msat}(c)$ is constructed as follows:

$$\begin{aligned}\mathrm{Msat}(c) &= \vee_{e\in\mathcal{E}}\, \exists(e,\ \mathrm{Msat}(e,c)) = \exists(\mathrm{id},\ \mathrm{Msat}(\mathrm{id},c)) \vee \exists(f,\ \mathrm{Msat}(f,c))\\ &= \exists(\mathrm{id},\ \exists(\bigcirc_1\ \bigcirc_2 \to \bigcirc_1 \to \bigcirc_2)) \vee \exists(f,\ \exists(\bigcirc_1 \to \bigcirc_1 \circlearrowright))\\ &\equiv \exists(\bigcirc_1\ \bigcirc_2 \to \bigcirc_1 \to \bigcirc_2) \vee \exists(\bigcirc_1\ \bigcirc_2 \to \bigcirc_{1=2} \circlearrowright)\end{aligned}$$

where $f\colon \bigcirc_1\ \bigcirc_2 \to \bigcirc_{1=2}$. The meaning of $\mathrm{Msat}(c)$ in case of $\mathcal{M}$-satisfiability is "There is a proper edge from the image of node 1 to a distinct image of node 2 or both images are identical and there is a loop".

Before we show Theorem 1, we prove a property for the auxiliary transformation $\mathrm{Msat}(e,c)$.

Lemma 1 ($\mathrm{Msat}(e,c)$). Let $\langle\mathcal{C},\mathcal{M}\rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object and epi-$\mathcal{M}$-factorization. For every condition c over P, every epimorphism $e\colon P \to P'$, and every morphism $p'\colon P' \to G$ in $\mathcal{M}$, $p' \models \mathrm{Msat}(e,c) \Leftrightarrow p' \circ e \models_{\mathcal{A}} c$.

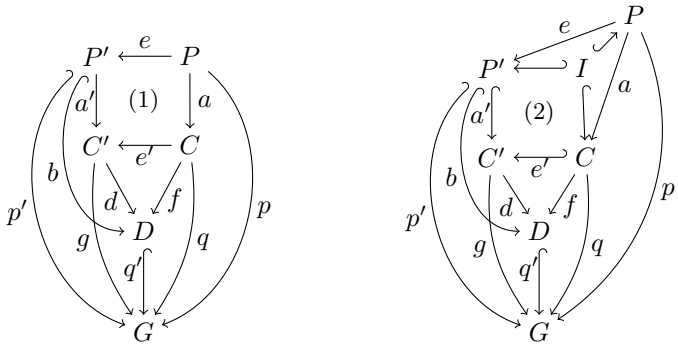


Proof. By structural induction.
Basis. For $c = \mathrm{true}$, we have $c = \mathrm{true} = \mathrm{Msat}(e, \mathrm{true}) = \mathrm{Msat}(e,c)$.
Hypothesis. Assume the statement holds for condition c'.
Step. For $c = \exists(a,c')$, we distinguish two cases:

Case 1. $\langle\mathcal{C},\mathcal{M}\rangle$ has pushouts of arbitrary morphisms. **Only if.** Let $p' \models \mathrm{Msat}(e, \exists(a,c')) = \vee_{d\in\mathcal{E}} \exists(b, \mathrm{Msat}(f,c'))$. There is some epimorphism $d\colon C' \to D$ with $b = d \circ a'$ in $\mathcal{M}$ and $f = d \circ e'$ such that $p' \models \exists(b, \mathrm{Msat}(f,c'))$. By definition of $\mathcal{M}$-satisfiability, there is some $q'\colon D \to G$ in $\mathcal{M}$ such that $q' \circ b = p'$ and $q' \models \mathrm{Msat}(f,c')$. Define now $q = q' \circ f$. Together with $q' \circ b = p'$, $b = d \circ a'$, $a' \circ e = e' \circ a$, $f = d \circ e'$, we observe $q \circ a = p$ ($p \models \exists a$). By inductive hypothesis, $q \models_{\mathcal{A}} c'$, therefore $p \models_{\mathcal{A}} \exists(a,c')$. **If.** Let $p \models_{\mathcal{A}} \exists(a,c')$. By definition of $\mathcal{A}$-satisfiability, there is some $q\colon C \to G$ such that $q \circ a = p$ and $q \models_{\mathcal{A}} c'$. Let (1) be the pushout of $a\colon P \to C$ and $e\colon P \to P'$. By the universal property of pushouts, there is some $g\colon C' \to G$ with $g \circ a' = p'$ and $g \circ e' = q$. Let $g = q' \circ d$ be an epi-$\mathcal{M}$-factorization of g with epimorphism d and monomorphism q' in $\mathcal{M}$, $b = d \circ a'$, and $f = d \circ e'$. Since p' and q' are in $\mathcal{M}$, $q' \circ b = p'$, and $\mathcal{M}$ is closed under decomposition, the morphism b is in $\mathcal{M}$ and d is in $\mathcal{E}$. As $p' = g \circ a'$, $g = q' \circ d$ and $b = d \circ a'$, we observe $p' = q' \circ b$ ($p' \models \exists b$). By inductive hypothesis, $q' \models \mathrm{Msat}(f,c')$, therefore $p' \models \vee_{d\in\mathcal{E}} \exists(b, \mathrm{Msat}(f,c')) = \mathrm{Msat}(e, \exists(a,c'))$.

Case 2. $\langle\mathcal{C},\mathcal{M}\rangle$ has only pushouts along $\mathcal{M}$-morphisms. **Only if.** As above, by using the morphism $f\colon C \to D$ and the inductive hypothesis. **If.** Let $p \models_{\mathcal{A}} \exists(a,c')$. By definition of $\mathcal{A}$-satisfiability, there is some $q\colon C \to G$ such that $q \circ a = p$ and $q \models_{\mathcal{A}} c'$. Since $\langle\mathcal{C},\mathcal{M}\rangle$ has pushouts along $\mathcal{M}$-morphisms, we can construct the pushout (2) of the $\mathcal{M}$-morphisms $i_C\colon I \to C$ and $i_{P'}\colon I \to P'$ leading to morphisms $a'\colon P' \to C'$ and $e'\colon C \to C'$. By the universal property of pushouts, there is some $g\colon C' \to G$ with $g \circ a' = p'$ and $g \circ e' = q$.

Let $g = q' \circ d$ be an epi-$\mathcal{M}$-factorization of g with epimorphism d and monomorphism q' in $\mathcal{M}$, $b = d \circ a'$, and $f = d \circ e'$. Since p' and q' are in $\mathcal{M}$, $q' \circ b = p'$, and $\mathcal{M}$ is closed under decomposition, the morphism b is in $\mathcal{M}$ and d is in $\mathcal{E}$. It turns out that $q' \circ b \circ e = q' \circ f \circ a$ and, by the monomorphism property of q', $b \circ e = f \circ a$. As b is in $\mathcal{M}$ and $f \circ a = b \circ e$, the tuple $\langle D, b, f \rangle$ belongs to the construction. As $p' = g \circ a'$, $g = q' \circ d$ and $b = d \circ a'$, we observe $p' = q' \circ b$ ($p' \models \vee_{d \in \mathcal{E}} \exists b$). By inductive hypothesis, $q' \models \mathrm{Msat}(f, c')$, therefore $p' \models \vee_{d \in \mathcal{E}} \exists(b, \mathrm{Msat}(f, c')) = \mathrm{Msat}(e, \exists(a, c'))$.



For Boolean formulas over conditions, the statement follows from the definitions and the inductive hypothesis. Consequently, the statement holds for all conditions. □

Theorem 1 follows directly from Lemma 1.

Proof of Theorem 1. Let $p\colon P \to G$ be a morphism and $p = p' \circ e$ an epi-$\mathcal{M}$-factorization of p with epimorphism e and monomorphism p' in $\mathcal{M}$. By Lemma 1, the definitions of $\mathcal{M}$-satisfiability, $\models$, and $\mathrm{Msat}(c)$ and the uniqueness of epi-$\mathcal{M}$-factorizations up to isomorphism, we obtain the statement for Msat: $p \models_{\mathcal{A}} c \Leftrightarrow p' \models \mathrm{Msat}(e, c) \Leftrightarrow p \models \exists(e,\ \mathrm{Msat}(e, c)) \Leftrightarrow p \models \vee_{e \in \mathcal{E}'} \exists(e,\ \mathrm{Msat}(e, c)) = \mathrm{Msat}(c)$. Fact 5 lifts the result to objects and conditions over I. □

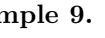
Remark. The case differentiation "$\langle \mathcal{C}, \mathcal{M} \rangle$ has pushouts" in the first construction of Theorem 1 is not necessary, if one assumes that the conditions are in $\mathcal{M}$-normal form. However, in the context of $\mathcal{A}$-satisfiability, this is a significant restriction, e.g., conditions like "the morphism is in $\mathcal{M}$" in the proof of Theorem 2 can only be expressed using morphisms not in $\mathcal{M}$.

Under certain assumptions, there is also a transformation from $\mathcal{M}$- to $\mathcal{A}$-satisfiability.

Theorem 2 (from $\mathcal{M}$- to $\mathcal{A}$-satisfiability). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with epi-$\mathcal{M}$-factorization and $\mathcal{M}$ strictly closed under decomposition. There is a transformation Asat on conditions such that, for every condition c and for every morphism p, $p \models_{\mathcal{A}} \mathrm{Asat}(c) \Leftrightarrow p \models c$, and for every condition c over I and for every object G, $G \models_{\mathcal{A}} \mathrm{Asat}(c) \Leftrightarrow G \models c$.

$\mathcal{A}$-satisfiable conditions allow elements to be identified. The idea of the construction is to prevent identification by expressing the property "the morphism is in $\mathcal{M}$" by subconditions.

Construction. For a morphism $a\colon P \to C$ and a condition c over C, $\mathrm{Asat}(\mathrm{true}) = \mathrm{true}$ and $\mathrm{Asat}(\exists(a, c')) = \exists(a,\ \mathrm{inM}_C \wedge \mathrm{Asat}(c'))$ where $\mathrm{inM}_C = \wedge_{e \in \mathcal{E}} \neg\exists e$ is a condition over C, the conjunction ranges over all epimorphisms $e\colon C \to C'$ not in $\mathcal{M}$. For Boolean formulas over conditions, the transformation is extended in the usual way.

Example 9. The condition $c = \exists($  $)$ meaning for graph morphisms w.r.t. $\mathcal{M}$-satisfiability "There exists a proper edge from the image of 1 to the distinct image of 2" is transformed into the condition $\mathrm{Asat}(c) = \exists($  $)$ meaning w.r.t. $\mathcal{A}$-satisfiability "There exists an edge from the image of 1 to the image of 2 and the images are distinct".

Proof. First, we prove that for every morphism $q\colon C \to G$, $q \models_{\mathcal{A}} \mathrm{inM}_C$ iff q is in $\mathcal{M}$. Proof by contraposition. **Only if.** Assume $q \models_{\mathcal{A}} \mathrm{inM}_C$, but q not in $\mathcal{M}$. Consider an epi-$\mathcal{M}$-factorization $q = q' \circ e$ of q with epimorphism e and monomorphism q' in $\mathcal{M}$. Then e is not in $\mathcal{M}$, $q \models_{\mathcal{A}} \exists e$ and $q \not\models_{\mathcal{A}} \mathrm{inM}_C$. Otherwise, by closure of $\mathcal{M}$ under compositions, e and q' in $\mathcal{M}$ would imply q in $\mathcal{M}$, contradiction. **If.** Assume $q \not\models_{\mathcal{A}} \mathrm{inM}_C$ and q in $\mathcal{M}$. Then $q \models_{\mathcal{A}} \exists e$ for some epimorphism $e\colon C \to C'$ not in $\mathcal{M}$. Then there is some $q'\colon C' \to G$ such that $q' \circ e = q$. Then q is not in $\mathcal{M}$. Otherwise, by the strict closure of $\mathcal{M}$ under decomposition, q in $\mathcal{M}$ would imply e in $\mathcal{M}$, contradiction.

$$\exists(\ C \xrightarrow{\ e\ } C'\ )$$
$$\qquad\quad q \searrow\ =\ \swarrow q'$$
$$\qquad\quad G$$

The statement for Asat is shown by structural induction:
Basis. For $c = \mathrm{true}$, we have $c = \mathrm{true} = \mathrm{Asat}(\mathrm{true}) = \mathrm{Asat}(c)$.
Hypothesis. Assume the statement holds for condition c'.
Step. For $c = \exists(a, c')$, we have the following. **If**. Let $p \models \exists(a, c')$. Then there is a morphism $q\colon C \to G$ in $\mathcal{M}$ such that $q \circ a = p$ and $q \models c'$. By the inductive hypothesis and the application condition inM_C being equivalent to "morphism is in $\mathcal{M}$", $q \models_{\mathcal{A}} \mathrm{inM}_C$ and $q \models_{\mathcal{A}} \mathrm{Asat}(c')$. Consequently, $p \models_{\mathcal{A}} \exists(a,\ \mathrm{inM}_C \wedge \mathrm{Asat}(c')) = \mathrm{Asat}(\exists(a, c'))$. **Only if.** Let $p \models_{\mathcal{A}} \mathrm{Asat}(\exists(a, c')) = \exists(a,\ \mathrm{inM}_C \wedge \mathrm{Asat}(c'))$. Then there is some $q\colon C \to G$ such that $q \circ a = p$, $q \models_{\mathcal{A}} \mathrm{inM}_C$, and $q \models_{\mathcal{A}} \mathrm{Asat}(c')$. The property of inM_C yields $q \in \mathcal{M}$, and by the inductive hypothesis, $q \models c'$. Together, $p \models \exists(a, c')$. For Boolean formulas over conditions, the statement follows from the definitions and the inductive hypothesis. This completes the inductive proof. Fact 5 lifts the result to objects and conditions over I. □

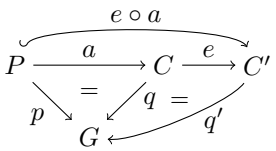
Fact 7. The construction above inserts the requirement "in $\mathcal{M}$" behind every morphism of the condition. However, it suffices to express that newly introduced elements are distinct. An optimized transformation Asat can be defined as follows:

$$\begin{aligned} \mathrm{Asat}(\mathrm{true}) &= \mathrm{true} & \qquad \mathrm{Asat}'(\mathrm{true}) &= \mathrm{true} \\ \mathrm{Asat}(\exists(a, c')) &= \exists(a,\ \mathrm{inM}_C \wedge \mathrm{Asat}'(c')) & \qquad \mathrm{Asat}'(\exists(a, c')) &= \exists(a,\ \mathrm{inM}_a \wedge \mathrm{Asat}'(c')) \end{aligned}$$

where $\mathrm{inM}_a = \wedge_{e \in \mathcal{E}'} \neg\exists e$ is a condition over C and $\mathcal{E}'$ is constructed as follows: Let $\mathcal{E}$ be set of all epimorphisms $e\colon C \to C'$ not in $\mathcal{M}$ such that $e \circ a$ in $\mathcal{M}$. The set $\mathcal{E}'$ consists of all epimorphisms $e \in \mathcal{E}$ such that there is no (non-isomorphic) decomposition of e into

epimorphisms $e'' \circ e' = e$ such that $e' \in \mathcal{E}$ and e'' not in $\mathcal{M}$. For all morphisms p in $\mathcal{M}$ and q with $p = q \circ a$, $q \models_{\mathcal{A}} \text{inM}_a$ iff q in $\mathcal{M}$. Note, for the initial $\mathcal{M}$-morphism $i_C \colon I \to C$, $\text{inM}_{i_C} = \text{inM}_C$.

Proof. We show for all morphisms p in $\mathcal{M}$ and q with $p = q \circ a$, $q \models_{\mathcal{A}} \text{inM}_a$ iff q in $\mathcal{M}$. By contraposition. Let p be in $\mathcal{M}$ and $p = q \circ a$.



Only if. Assume $q \models_{\mathcal{A}} \text{inM}_a$, but q not in $\mathcal{M}$. As in the proof of Theorem 2, consider an epi-$\mathcal{M}$-factorization $q = q' \circ e$ of q with epimorphism e and monomorphism q' in $\mathcal{M}$. Then e is not in $\mathcal{M}$. Otherwise, by closure of $\mathcal{M}$ under compositions, e and q' in $\mathcal{M}$ would imply q in $\mathcal{M}$. Moreover, $e \circ a$ in $\mathcal{M}$. Otherwise, by q' in $\mathcal{M}$ and the closure of $\mathcal{M}$ under decompositions, $e \circ a$ not in $\mathcal{M}$ would imply p not in $\mathcal{M}$. Now e is an epimorphism not in $\mathcal{M}$ and $e \circ a$ in $\mathcal{M}$, therefore $e \in \mathcal{E}$. If there is now a decomposition $e'' \circ e'$ of e with $e' \in \mathcal{E}$ and e'' not in $\mathcal{M}$, let $e = e'$ from now and repeat this argument. If there is no such decomposition, we have $e \in \mathcal{E}'$, and we conclude $q \models_{\mathcal{A}} \exists e$ and $q \not\models_{\mathcal{A}} \text{inM}_a$, contradiction.
If. Assume $q \not\models_{\mathcal{A}} \text{inM}_a$ and q in $\mathcal{M}$. As in the proof of Theorem 2, $q \not\models_{\mathcal{A}} \text{inM}_a$ implies q not in $\mathcal{M}$, contradiction. □

By Theorems 1 and 2, we obtain the following corollary.

Corollary 1 (equivalence). For weak adhesive HLR categories with $\mathcal{M}$-initial object, epi-$\mathcal{M}$-factorization, and $\mathcal{M}$ strictly closed under decomposition, $\mathcal{A}$-satisfiability and $\mathcal{M}$-satisfiability are expressively equivalent.

$$\mathcal{A}\text{-satisfiability} \underset{\text{Asat}}{\overset{\text{Msat}}{\rightleftarrows}} \mathcal{M}\text{-satisfiability}$$

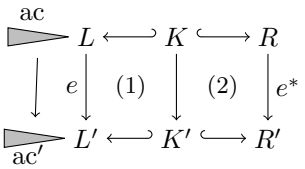
The equivalence result is valid for nested constraints and application conditions; it is not valid for plain constraints in the sense of (Ehrig et al. 2006b), as the presented transformations increase the depth of nesting. As shown later in Section 8, the transformations Msat and Asat are an important step in the conversion of graph conditions into first-order graph formulas and vice versa.

5. Matching of rules

In this section, we investigate the different matching notions for rules and show that $\mathcal{A}$- and $\mathcal{M}$-matching are expressively equivalent. First, there is a transformation from $\mathcal{A}$- to $\mathcal{M}$-matching. We establish a Simulation Theorem saying that any direct derivation with arbitrary matching can be simulated by a direct derivation with $\mathcal{M}$-matching. This extends the Simulation Theorem in (Habel et al. 2001) to weak adhesive HLR categories

with epi-$\mathcal{M}$-factorizations, in which a single rule is simulated by set of so-called quotient rules.

More precisely, we introduce quotient constructions that transform plain rules and conditions with respect to epimorphisms, e.g. epimorphism e as depicted below. To support both satisfiability notions of conditions independently of the matching notion, we consider two quotient constructions for conditions, N and B.



If the application condition ac is $\mathcal{M}$-satisfiable, $\mathrm{ac}' = \mathrm{N}(e, \mathrm{ac})$, while $\mathrm{ac}' = \mathrm{B}(e, \mathrm{ac})$ in the case of $\mathcal{A}$-satisfiability.

Lemma 2 (quotients of conditions). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with epi-$\mathcal{M}$-factorization. There is a transformation N such that, for all conditions c over P, and all morphisms p with domain P and epi-$\mathcal{M}$-factorization $p = p' \circ e$,

$$p' \models \mathrm{N}(e, c) \iff p \models c.$$

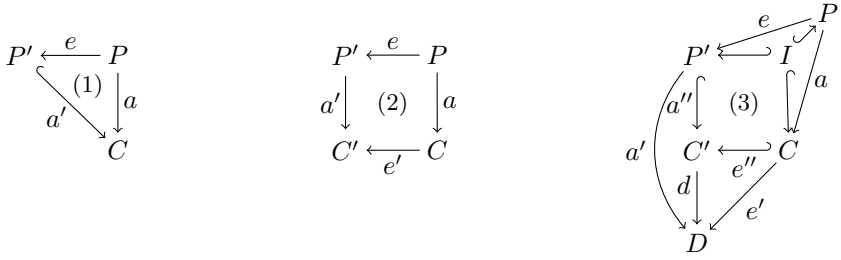
There is a transformation B such that, for all conditions c over P, all morphisms p with domain P, and all factorizations $p = p' \circ e$ (e is not required to be an epimorphism),

$$p' \models_{\mathcal{A}} \mathrm{B}(e, c) \iff p \models_{\mathcal{A}} c.$$

Construction. $\mathrm{N}(e, \mathrm{true}) = \mathrm{true}$ and $\mathrm{N}(e, \exists(a, c')) = \exists(a', c')$ if e can be extended to an epi-$\mathcal{M}$-factorization of a, i.e. if there exists a morphism a' in $\mathcal{M}$ such that $a = a' \circ e$, otherwise $\mathrm{N}(e, \exists(a, c')) = \mathrm{false}$.

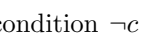
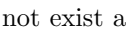
If $\langle \mathcal{C}, \mathcal{M} \rangle$ has pushouts of arbitrary morphisms, let $\mathrm{B}(e, \mathrm{true}) = \mathrm{true}$ and $\mathrm{B}(e, \exists(a, c')) = \exists(a', \mathrm{B}(e', c'))$ where (2) is the pushout of the morphisms a and e leading to the morphisms a' and e'.

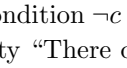
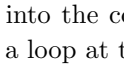
Alternatively, let $\mathrm{B}(e, \mathrm{true}) = \mathrm{true}$ and $\mathrm{B}(e, \exists(a, c')) = \vee_{d \in \mathcal{E}} \exists(a', \mathrm{B}(e', c'))$ and construct the pushout (3) of the initial $\mathcal{M}$-morphisms $i_{P'}$ and i_C yielding the morphisms a'' and e'' to object C', let the set $\mathcal{E}$ range over all epimorphisms $d\colon C' \to D$ such that $d \circ e'' \circ a = d \circ a'' \circ e$ and define $a' = d \circ a''$ and $e' = d \circ e''$.



For Boolean formulas over conditions, the transformations are extended in the usual way.

When the existence of pushouts of arbitrary morphisms cannot be guaranteed, transformation B preserves the logical structure and therefore the size of the condition. In case the existence of pushouts cannot be guaranteed, transformation B is similar to the alternative construction of transformation Msat of Theorem 1, except that a' is arbitrary and not required to be in $\mathcal{M}$, corresponding to just some epi-factorization and not necessarily an epi-$\mathcal{M}$-factorization.

Example 10. The condition $\neg c = \neg\exists($  $)$ meaning for graph morphisms w.r.t. $\mathcal{A}$ [$\mathcal{M}$]-satisfiability "There does not exist a [proper] edge from the image of 1 to the image of 2" is transformed by the surjective graph morphism $e:$  into the condition $\mathrm{B}(e,\neg c) = \neg\mathrm{B}(e,c)) = \neg\exists($  $)$ [$\mathrm{N}(e,\neg c) = \neg\mathrm{N}(e,c) = \neg\mathrm{false} = \mathrm{true}$] meaning "There does not exist a loop at the common image of node 1 and 2" ["Always satisfied"].

Example 11. The condition $\neg c = \neg\exists($  $)$ meaning for graph morphisms w.r.t. $\mathcal{A}$- and $\mathcal{M}$-satisfiability "There does not exist a loop at the common image of 1 and 2" is transformed over e into the condition $\mathrm{B}(e,\neg c) = \neg\exists($  $) = \mathrm{N}(e,\neg c)$ meaning "There does not exist a loop at the image of the node".

Proof. By structural induction.

Case 1. Transformation N.

Basis. For $c = \mathrm{true}$, we have $c = \mathrm{true} = \mathrm{N}(e,\mathrm{true}) = \mathrm{N}(e,c)$.

Hypothesis. Assume the statement holds for condition c'.

Step. For $c = \exists(a,c')$, we have the following. **Only if.** Let $p' \models \mathrm{N}(e,\exists(a,c'))$. As no morphism satisfies false, $\mathrm{N}(e,\exists(a,c')) = \exists(a',c')$ for some morphism a' in $\mathcal{M}$ with epi-$\mathcal{M}$ factorization $a = a' \circ e$. Then there is some $q\colon C \to G$ in $\mathcal{M}$ such that $q \circ a' = p'$ and $q \models c'$. Moreover, $q \circ a = p$. Consequently, $p \models \exists(a,c')$. **If.** Let $p \models \exists(a,c')$. Then there is some $q\colon C \to G$ in $\mathcal{M}$ such that $q \circ a = p$ and $q \models c'$. Let $a = a'' \circ e'$ be an epi-$\mathcal{M}$-factorization of a with epimorphism $e'\colon P \to P''$ and monomorphism $a''\colon P'' \to C$ in $\mathcal{M}$. Then $p'' = q \circ a''$ is in $\mathcal{M}$. Since $a = a'' \circ e'$ is an epi-$\mathcal{M}$-factorization of a, $p = q \circ a = q \circ a'' \circ e' = p'' \circ e'$ is an epi-$\mathcal{M}$-factorization of p with epimorphism e' and monomorphism p'' in $\mathcal{M}$. As $p = p' \circ e'$ is also an epi-$\mathcal{M}$-factorization, the uniqueness of epi-$\mathcal{M}$-factorizations yields $e = e'$, $p' = p''$, and $a' = a''$ (up to isomorphism). Therefore, $p' \models \exists(a',c')$.

Case 2. Transformation B, $\langle\mathcal{C},\mathcal{M}\rangle$ has pushouts of arbitrary morphisms.

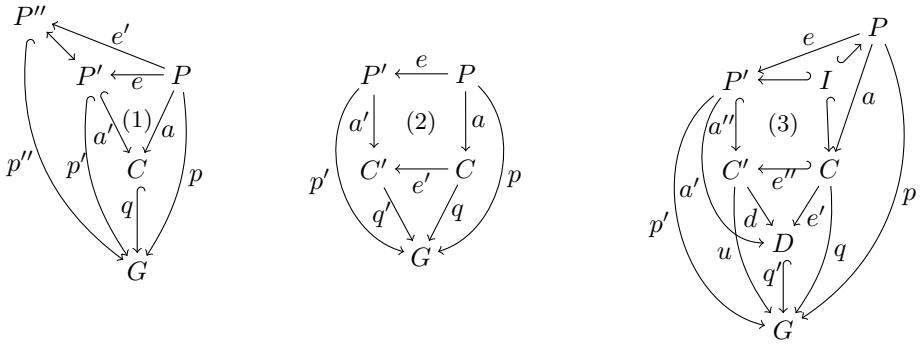
Basis. For $c = \mathrm{true}$, we have $c = \mathrm{true} = \mathrm{B}(e,\mathrm{true}) = \mathrm{B}(e,c)$.

Hypothesis. Assume the statement holds for condition c'.

Step. For $c = \exists(a,c')$, we have the following. **Only if.** Let $p' \models_{\mathcal{A}} \mathrm{B}(e,\exists(a,c')) = \exists(a',\mathrm{B}(e',c'))$. Then there is some $q'\colon C' \to G$ such that $q' \circ a' = p'$ and $q' \models_{\mathcal{A}} \mathrm{B}(e',c')$. By inductive hypothesis, $q' \circ e' \models_{\mathcal{A}} c'$. Define $q = q' \circ e'$. Then $q \circ a = q' \circ e' \circ a = p' \circ e = p$. Consequently, $p \models_{\mathcal{A}} \exists(a,c')$. **If.** Let $p \models_{\mathcal{A}} \exists(a,c')$. Then there is some $q\colon C \to G$ such that $q \circ a = p$ and $q \models_{\mathcal{A}} c'$. Construct pushout (2). By the universal property of pushouts, there is a morphism $q'\colon C' \to G$ such that $q' \circ a' = p'$ and $q' \circ e' = q$. By inductive hypothesis, $q' \models_{\mathcal{A}} \mathrm{B}(e',c')$. Consequently, $p' \models_{\mathcal{A}} \exists(a',\mathrm{B}(e',c')) = \mathrm{B}(e,\exists(a,c'))$.

Case 3. Transformation B, $\langle\mathcal{C},\mathcal{M}\rangle$ has an $\mathcal{M}$-initial object. The proof is similar to

that of Msat of Theorem 1.
Basis. For $c = \text{true}$, we have $c = \text{true} = \text{B}(e, \text{true}) = \text{B}(e, c)$.
Hypothesis. Assume the statement holds for condition c'.
Step. For $c = \exists(a, c')$, we have the following. **Only if.** As in Case 2, by case differentiation for some $d\colon C' \to D$, using the corresponding morphism $e'\colon C \to D$ and the inductive hypothesis. **If**. Let $p \models_{\mathcal{A}} \exists(a, c')$. By definition of $\mathcal{A}$-satisfiability, there is some $q\colon C \to G$ such that $q \circ a = p$ and $q \models_{\mathcal{A}} c'$. Since $\langle \mathcal{C}, \mathcal{M} \rangle$ has pushouts along $\mathcal{M}$-morphisms, we can construct the pushout (3) of the $\mathcal{M}$-morphisms $i_C\colon I \to C$ and $i_{P'}\colon I \to P'$ leading to morphisms $a''\colon P' \to C'$ and $e''\colon C \to C'$. By the universal property of pushouts, there is some $u\colon C' \to G$ with $u \circ a'' = p'$ and $u \circ e'' = q$. Let $u = q' \circ d$ be an epi-$\mathcal{M}$-factorization of u with epimorphism d and monomorphism q' in $\mathcal{M}$, define $a' = d \circ a''$ and $e' = d \circ e''$. It turns out that $q' \circ a' \circ e = q' \circ e' \circ a$ and, by the monomorphism property of q', $a' \circ e = e' \circ a$. Therefore, d belongs to the set $\mathcal{E}$. As $p' = u \circ a''$, $u = q' \circ d$ and $a' = d \circ a''$, we have $p' = q' \circ a'$ and $p' \models \exists a'$. By inductive hypothesis, $q' \models_{\mathcal{A}} \text{B}(e', c')$. Consequently, $p' \models_{\mathcal{A}} \vee_{d \in \mathcal{E}} \exists(a', \text{B}(e', c')) = \text{B}(e, \exists(a, c'))$.



For Boolean formulas over conditions, the statement follows from the definitions and the inductive hypothesis. Consequently, the statement holds for all conditions. □

Remark. In case of the alternative construction of transformation B, i.e. $\text{B}(e, \text{true}) = \text{true}$ and $\text{B}(e, \exists(a, c')) = \vee_{d \in \mathcal{E}} \exists(a', \text{B}(e', c'))$, the set $\mathcal{E}$ may be restricted to a set $\mathcal{E}'$ with $d \in \mathcal{E}'$, iff $d \in \mathcal{E}$ and there exist no $d' \in \mathcal{E}$ and no epimorphism f not in $\mathcal{M}$, such that $d = d' \circ f$ (d is more general than d'). In case of pushouts and $\mathcal{M}$-initial objects, this optimized transformation yields eventually the only one morphism d with the pushout object of (2) as its codomain.

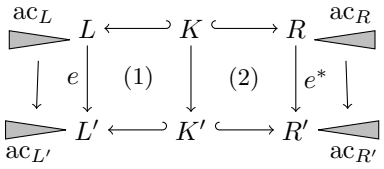
Now we are able to present a transformation from $\mathcal{A}$- to $\mathcal{M}$-matching. For simplicity, we assume a fixed satisfiability notion. For a rule ρ, let $[\![\rho]\!]_{\mathcal{A}}$ and $[\![\rho]\!]_{\mathcal{M}}$ denote the sets of pairs $\langle G, H \rangle$ with direct derivation $G \Rightarrow_{\rho, m, m^*} H$ through a match m in $\mathcal{A}$ and $\mathcal{M}$, respectively. The definition is extended to sets of rules $\mathcal{R}$, i.e. $[\![\mathcal{R}]\!]_{\mathcal{A}} = \bigcup_{\rho \in \mathcal{R}} [\![\rho]\!]_{\mathcal{A}}$ and $[\![\mathcal{R}]\!]_{\mathcal{M}} = \bigcup_{\rho \in \mathcal{R}} [\![\rho]\!]_{\mathcal{M}}$.

Theorem 3 (from $\mathcal{A}$- to $\mathcal{M}$-matching). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with epi-$\mathcal{M}$-factorization, and, in the case of $\mathcal{A}$-satisfiability, pushouts or an $\mathcal{M}$-initial

object. There is a transformation Q, such that for every rule ρ,

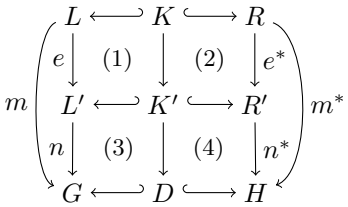
$$[\![\rho]\!]_{\mathcal{A}} = [\![\mathrm{Q}(\rho)]\!]_{\mathcal{M}}.$$

Construction. For a rule $\rho = \langle q, \mathrm{ac}\rangle$ with $\mathrm{ac} = \langle \mathrm{ac}_L, \mathrm{ac}_R\rangle$, let $\mathrm{Q}(\rho)$ be the set of rules $\rho' = \langle q', \mathrm{ac}'\rangle$ where q' is a *quotient rule* of q with respect to an epimorphism e, i.e., there are two pushouts (1) and (2) with match e and comatch e^* as below, and $\mathrm{ac}' = \langle \mathrm{N}(e, \mathrm{ac}_L), \mathrm{N}(e^*, \mathrm{ac}_R)\rangle$ for $\mathcal{M}$-satisfiability and $\langle \mathrm{B}(e, \mathrm{ac}_L), \mathrm{B}(e^*, \mathrm{ac}_R)\rangle$ otherwise.



Example 12. Consider the rule $\langle \bigcirc_1\ \bigcirc_2 \hookleftarrow \bigcirc_1\ \bigcirc_2 \hookrightarrow \bigcirc_1 \to \bigcirc_2\rangle$ with left application condition $\mathrm{ac} = \neg\exists(\bigcirc_1\ \bigcirc_2 \to \bigcirc_1 \to \bigcirc_2)$ meaning w.r.t. $\mathcal{A}$-matching and $\mathcal{A}$-satisfiability "Add an edge, provided there does not exist one". Then the rule $\langle \bigcirc \hookleftarrow \bigcirc \hookrightarrow \bigcirc\!\!\circlearrowright\rangle$ with left application condition $\mathrm{B}(\bigcirc\ \bigcirc \to \bigcirc, \mathrm{ac}) = \neg\exists(\bigcirc \to \bigcirc\!\!\circlearrowright)$ (compare with Example 10) is a quotient rule meaning w.r.t. $\mathcal{M}$-matching "Add a loop at the node, provided there does not exist one".

Proof. **If**. Let $G \Rightarrow_{\rho',n,n^*} H$ be a direct derivation through $\rho' = \langle q', \mathrm{ac}'\rangle$ with ac' as in the construction and $n \in \mathcal{M}$. Then the diagrams (1), (2), (3) and (4) in the figure below are pushouts and, by the Composition Lemma of pushouts, the composed diagrams (1)+(3) and (2)+(4) are pushouts as well. Hence, there is a direct derivation $G \Rightarrow_{q,m,m^*} H$ with $m \in \mathcal{A}$. By assumption, $n \models \mathrm{ac}'_L$ and $n^* \models \mathrm{ac}'_R$ ($\mathcal{M}$-satisfiability) or $n \models_{\mathcal{A}} \mathrm{ac}'_L$ and $n^* \models_{\mathcal{A}} \mathrm{ac}'_R$ ($\mathcal{A}$-satisfiability). By Lemma 2, $m \models \mathrm{ac}_L$ and $m^* \models \mathrm{ac}_R$. Thus, $G \Rightarrow_{\rho,m,m^*} H$ with $m \in \mathcal{A}$.



Only if. Let $G \Rightarrow_{\rho,m,m^*} H$ be a direct derivation through $\rho = \langle q, \mathrm{ac}\rangle$ with $m \in \mathcal{A}$, and let $m = n \circ e$ be an epi-$\mathcal{M}$ factorization of m with epimorphism $e\colon L \to L'$ and monomorphism $n\colon L' \to G$ in $\mathcal{M}$. Then there is a decomposition of the original diagrams into diagrams (1) and (3), (2) and (4) as follows: Construct the object K' as a pullback object of $L' \to G \leftarrow D$ and denote the diagram by (3). By the universal property of pullbacks, there is a unique morphism $e'\colon K \to K'$ such that $K \to K' \to D = K \to D$ and diagram (1) commutes. By the pushout-pullback decomposition, (1) and (3) are pushouts. Now construct the object R' as the pushout object of $K' \leftarrow K \to R$ and denote the diagram by (2). By the universal property of pushouts, there is a unique

morphism $n^*\colon R' \to H$ such that $R \to R' \to H = R \to H$ and diagram (4) commutes. By the Decomposition Lemma for pushouts, diagram (4) is a pushout. Since epimorphisms and $\mathcal{M}$-morphisms are closed under pullbacks and pushouts, the vertical morphisms e, e', and e^* are epimorphisms and n, n', and n^* are in $\mathcal{M}$. Let ac' as in the construction. Then $\rho' = \langle q', \mathrm{ac}' \rangle$ is a rule in $\mathrm{Q}(p)$ and, by Lemma 2, $G \Rightarrow_{\rho',n,n^*} H$ is a direct derivation with $n \in \mathcal{M}$. □

If $\mathcal{M}$ strictly closed under decomposition, there is a transformation from $\mathcal{M}$- to $\mathcal{A}$-matching.

Theorem 4 (from $\mathcal{M}$- to $\mathcal{A}$-matching). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category and $\mathcal{M}$ strictly closed under decomposition. There is a transformation R such that, for every rule ρ,

$$[\![\rho]\!]_{\mathcal{M}} = [\![\mathrm{R}(\rho)]\!]_{\mathcal{A}}.$$

The idea of the transformation is to use the application condition inM which is satisfied iff the match is in $\mathcal{M}$.

Construction. For a rule $\rho = \langle q, \mathrm{ac} \rangle$, let $\mathrm{R}(\rho) = \langle q, \langle \mathrm{inM}_L \wedge \mathrm{ac}_L, \mathrm{ac}_R \rangle \rangle$.

Example 13. The rule $q = \langle \bigcirc_1 \ \bigcirc_2 \hookleftarrow \bigcirc_1 \ \bigcirc_2 \hookrightarrow \bigcirc_1 \to \bigcirc_2 \rangle$ with left application condition $\mathrm{ac}_L = \neg\exists(\bigcirc_1 \ \bigcirc_2 \to \bigcirc_1 \to \bigcirc_2)$, meaning w.r.t. $\mathcal{M}$-matching "Add an edge between distinct nodes, provided there does not exist a connecting edge", is transformed into the rule q with left application condition $\mathrm{inM}_L \wedge \mathrm{ac}_L = \neg\exists(\bigcirc_1 \ \bigcirc_2 \to \bigcirc_{1=2}) \wedge \neg\exists(\bigcirc_1 \ \bigcirc_2 \to \bigcirc_1 \to \bigcirc_2)$ meaning w.r.t. $\mathcal{A}$-matching "Add an edge between the nodes, provided the nodes are distinct and there does not exist a connecting edge".

Proof. Since $\mathcal{M}$ is strictly closed under decomposition, the property "the match is in $\mathcal{M}$" can be expressed by the application condition inM (see the first statement in the proof of Theorem 2). Then, for every rule ρ, $G \Rightarrow_{\mathcal{M},\rho} H$ iff $G \Rightarrow_q H$ for some $q \in \mathrm{R}(\rho)$, i.e. $[\![\rho]\!]_{\mathcal{M}} = [\![\mathrm{R}(\rho)]\!]_{\mathcal{A}}$. □

By Theorems 3 and 4, we obtain the following corollary.

Corollary 2 (equivalence). For weak adhesive HLR categories with pushouts or an $\mathcal{M}$-initial object, epi-$\mathcal{M}$-factorization, and $\mathcal{M}$ strictly closed under decomposition, $\mathcal{A}$-matching and $\mathcal{M}$-matching are expressively equivalent.

$$\mathcal{A}\text{-matching} \underset{\mathrm{R}}{\overset{\mathrm{Q}}{\rightleftarrows}} \mathcal{M}\text{-matching}$$

6. Transformations of conditions

In the following, we present transformations from constraints to application conditions, from right- to left application conditions, and from application conditions to constraints.

Moreover, we consider the construction of explicit gluing conditions. These basic transformations can be used in the development of correct transformation systems, as we show in Section 7.

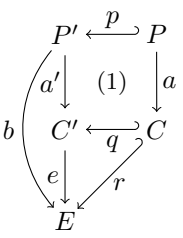
First, we show that a constraint can be transformed into an equivalent right application condition in the sense that the existence of a comatch is conjunctively incooperated into the constraint. This transformation is used in Section 7, e.g. in the construction of weakest preconditions. The construction is a generalized version of the corresponding construction for basic constraints in (Ehrig et al. 2006b), first described in (Heckel and Wagner 1995).

Theorem 5 (transformation of constraints into application conditions). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object I and epi-$\mathcal{M}$-factorization. There is a transformation A such that, for all conditions c over I, all rules ρ with right-hand side R, and all morphisms $m^*\colon R \to H$,

$$m^* \models \mathrm{A}(\rho, c) \iff H \models c.$$

The idea of the transformation A is to consider a disjunction of all possible overlappings of R and the objects of the condition.

Construction. For conditions c over I and rules ρ, define $\mathrm{A}(\rho, c) = \mathrm{A}_1(i_R, c)$, where i_R is the initial $\mathcal{M}$-morphism to R. For morphisms $p\colon P \to P'$ in $\mathcal{M}$, conditions over P and switch $k \in \{1, 2\}$,



$$\begin{aligned} \mathrm{A}_k(p,\ \mathrm{true}) &= \mathrm{true}, \\ \mathrm{A}_k(p,\ \exists(a, c')) &= \vee_{e \in \mathcal{E}}\, \exists(b,\ \mathrm{A}_2(r, c')). \end{aligned}$$

For a morphism $p\colon P \to P'$ in $\mathcal{M}$, construct the pushout (1) of p and a leading to morphisms $a'\colon P' \to C'$ and $q\colon C \to C'$. The disjunction $\vee_{e \in \mathcal{E}}$ ranges over all epimorphisms $e\colon C' \to E$ such that both $b = e \circ a'$ and $r = e \circ q$ are in $\mathcal{M}$, if $k = 2$, and just $r = e \circ q$ in $\mathcal{M}$, if $k = 1$.

For Boolean formulas over conditions, the transformation $\mathrm{A}_k(p, c)$ is extended in the usual way.

Example 14. Given the rule $\rho = \langle \bigcirc\ \bigcirc \hookleftarrow \bigcirc\ \bigcirc \hookrightarrow \bigcirc\!\!\rightarrow\!\!\bigcirc \rangle$, the graph constraint $c = \forall(\emptyset \to \bigcirc,\ \exists(\bigcirc \to \bigcirc\!\!\circlearrowright))$ meaning "Every node must have a loop" is transformed into the right application condition $\mathrm{A}(\rho, c) = \neg\vee_{j=1}^{5} \exists(\bigcirc\!\!\rightarrow\!\!\bigcirc \to P'_j,\ \neg\vee_\ell \exists(P'_j \to E_{j\ell})) = \wedge_{j=1}^{5} \forall(\bigcirc\!\!\rightarrow\!\!\bigcirc \to P'_j,\ \vee_\ell \exists(P'_j \to E_{j\ell}))$ for the graphs $P'_j, E_{j\ell}$ as in Figure 3, meaning "Every node (outside and inside the occurrence of the right-hand side) must have a loop".

Remark. For the initial $\mathcal{M}$-morphisms i_R, i_P, the construction of $\mathrm{A}_1(i_R, \exists i_P)$ and the first invocation of $\mathrm{A}_1(i_R, \exists(i_P, c))$ corresponds to the construction of all possible gluings $\langle E, b, r \rangle$ of the right-hand side R of the rule and the object P of the condition, such that the morphism r is in $\mathcal{M}$, as in (Ehrig et al. 2006b).

Before we prove Theorem 6, we state a general property of the transformation $\mathrm{A}_k(p, c)$. The following lemma states that a condition c over P can be shifted over a morphism p

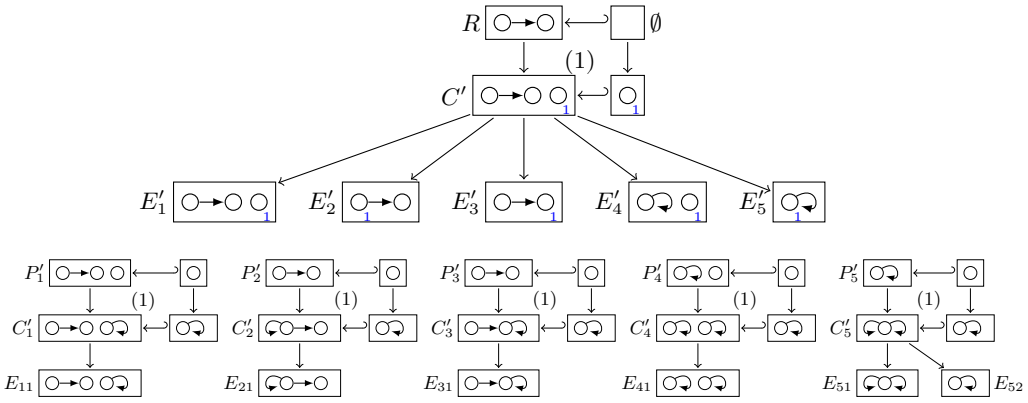
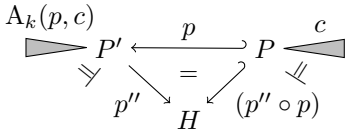


Figure 3. Transformation of a graph constraint into an application condition

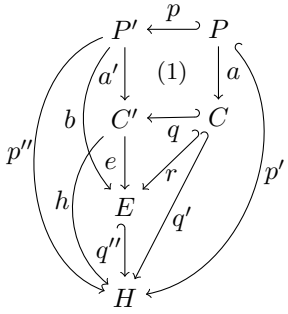
in $\mathcal{M}$ with $\mathrm{Dom}(p) = P$ in the sense of a conjunctive combination of c and $\exists p$, as used in (Pennemann 2008b) for instance.

Lemma 3 (transformation $\mathrm{A}_k(p, c)$). For all conditions c over P, all morphisms $p\colon P \to P'$ in $\mathcal{M}$ and $p''\colon P' \to H$ and either ($k = 1$ and p $\mathcal{M}$-initial) or (k=2 and p'' in $\mathcal{M}$), we have:

$$p'' \models \mathrm{A}_k(p, c) \iff p'' \circ p \models c.$$



Proof. By structural induction.
Basis. For $c = \mathrm{true}$, we have $c = \mathrm{true} = \mathrm{A}_k(p, \mathrm{true}) = \mathrm{A}_k(p, c)$.
Hypothesis. Assume the statement holds for condition c'.
Step. Let $c = \exists(a, c')$.



Only if. Assume $p'' \models \mathrm{A}_k(p, \exists(a, c')) = \vee_{e\in\mathcal{E}} \exists(b, \mathrm{A}_2(r, c'))$. There is an $e \in \mathcal{E}$ such that $p'' \models \exists(b, \mathrm{A}_2(r, c'))$. By definition of $\models$, there exists a morphism $q''\colon E \to H$ in $\mathcal{M}$ with $p'' = q'' \circ b$. Define $q' = q'' \circ r$. As q'', r in $\mathcal{M}$ and $\mathcal{M}$ closed under composition, q' in $\mathcal{M}$. Let $p' = p'' \circ p$. Either $p = i_R$ and $p' = i_H$ in $\mathcal{M}$ ($k = 1$) or p, p'' in $\mathcal{M}$, $\mathcal{M}$ closed under composition and p' in $\mathcal{M}$ ($k = 2$). By construction $a' \circ p = q \circ a$ ((1) is pushout), $r = e \circ q$ and $b = e \circ a'$. Together, $p'' \circ p = p' = q' \circ a$ ($p' = p'' \circ p \models \exists a$). Using the inductive hypothesis, $q'' \models \mathrm{A}_2(p, \exists(a, c'))$ implies $q' = q'' \circ r \models c'$, we have $p' = p'' \circ p \models \exists(a, c')$.

If. Assume $p'' \circ p \models \exists(a, c')$. Let $p' = p'' \circ p$. Either $p = i_R$ and $p' = i_H$ in $\mathcal{M}$ ($k = 1$) or p'', p in $\mathcal{M}$, $\mathcal{M}$ closed under composition and p' in $\mathcal{M}$ ($k = 2$). By definition of $\models$, there exists

a morphism $q' : C \to H$ in $\mathcal{M}$ with $p' = q' \circ a$. Following the construction, we yield the pushout (1) with object C' together with the morphisms a' and q. As $q' \circ a = p' = p'' \circ p$, the pushout guarantees the existence of a unique morphism $h : C' \to H$ with $p'' = h \circ a'$ and $q' = h \circ q$. Consider $q'' \circ e = h$, an epi-$\mathcal{M}$-factorization of h with epimorphism e and $\mathcal{M}$-morphism q''. Let $r = e \circ q$. As $q'' \circ r = q'$, q', q'' in $\mathcal{M}$ and $\mathcal{M}$ closed under decomposition, r in $\mathcal{M}$. Define $b = e \circ a'$. For $k = 1$, we are ready to conclude. For $k = 2$, the additional assumption p'' in $\mathcal{M}$ and p'', q'' in $\mathcal{M}$ and $\mathcal{M}$ closed under decomposition yields the additional requirement b in $\mathcal{M}$. In every case, $p'' = h \circ a'$, $h = q'' \circ e$ and $b = e \circ a'$ yield $p'' = q'' \circ b$ ($p'' \models \vee_{e \in \mathcal{E}} \exists b = \mathrm{A}_k(p, \exists a)$). Using the inductive hypothesis, $q' = q'' \circ r \models c'$ implies $q'' \models \mathrm{A}_2(p, \exists(a, c'))$, we have $p'' \models \vee_{e \in \mathcal{E}} \exists(b,\ \mathrm{A}_2(r, c')) = \mathrm{A}_k(p, \exists(a, c'))$.

For Boolean formulas over conditions, the statement follows directly from the definitions and the inductive hypothesis. Thus, the statement holds for all conditions over I. □

Proof of Theorem 5. According to Lemma 3, for all conditions c over I, all morphisms $i_R : I \to R$ in $\mathcal{M}$ and $m^* : R \to H$, we have: $m^* \models \mathrm{A}_1(i_R, c)$ iff $m^* \circ i_R \models c$ iff $i_H \models c$ iff $H \models c$. □

In the modified cases of $\mathcal{A}$-matching and $\mathcal{A}$-satisfiability or $\mathcal{M}$-matching and $\mathcal{M}$-satisfiability, there are simplified transformations.

Remark ($\mathcal{M}$-matching and $\mathcal{M}$-satisfiability). In the case of $\mathcal{M}$-matching and $\mathcal{M}$-satisfiability, one may directly use transformation $\mathrm{A}_2(i_R, c)$such that for all conditions c over I, all rules $\rho = \langle L \hookleftarrow K \hookrightarrow R \rangle$ we have: For all morphisms $m^* : R \to H$ in $\mathcal{M}$, $m^* \models \mathrm{A}_2(i_R, c) \Leftrightarrow H \models c$.

According to Lemma 3, for all conditions c over I, all morphisms $i_R : I \to R$ in $\mathcal{M}$ and $m^* : R \to H$ in $\mathcal{M}$, we have: $m^* \models \mathrm{A}_2(i_R, c)$ iff $m^* \circ i_R \models c$ iff $i_H \models c$ iff $H \models c$.

Remark ($\mathcal{A}$-matching and $\mathcal{A}$-satisfiability). In the case of $\mathcal{A}$-matching and $\mathcal{A}$-satisfiability, one may use transformation B of Lemma 2 such that for all conditions c over I, all rules $\rho = \langle L \hookleftarrow K \hookrightarrow R \rangle$ we have: For all morphisms $m^* : R \to H$, $m^* \models_{\mathcal{A}} \mathrm{B}(i_R, c) \Leftrightarrow H \models_{\mathcal{A}} c$.

For all conditions c over I, all initial $\mathcal{M}$-morphisms i_H, and all factorizations $i_H = m^* \circ i_R$ of morphism i_H, $m^* \models_{\mathcal{A}} \mathrm{B}(i_R, c) \Leftrightarrow i_H \models_{\mathcal{A}} c \Leftrightarrow H \models_{\mathcal{A}} c$.

With the help of transformation $\mathrm{N}(e, c)$ of Lemma 2, we can generalize transformation $\mathrm{A}_k(p, c)$ from morphisms p in $\mathcal{M}$ to arbitrary morphisms.

Corollary 3 (shifting of conditions over morphisms). There is a transformation Shift_k such that, for all conditions c over P and all morphisms $p : P \to P'$, $p'' : P' \to H$, and either (k=1 and p $\mathcal{M}$-initial) or (k=2 and p'' in $\mathcal{M}$),

$$p'' \models \mathrm{Shift}_k(p, c) \iff p'' \circ p \models c.$$

Construction. Let $p = m \circ e$ be a epi-$\mathcal{M}$-factorization of m into an epimorphism e and morphism m in $\mathcal{M}$. If k=1 and p $\mathcal{M}$-initia, then $e = \mathrm{id}_I$, p in $\mathcal{M}$ and let $\mathrm{Shift}_1(m, c) = \mathrm{A}_1(p, c)$. If k=2 and p'' in $\mathcal{M}$, then $\mathrm{Shift}_2(p, c) = \mathrm{A}_2(m, \mathrm{N}(e, c))$.

Proof. For k=2 and p'' in $\mathcal{M}$, the proof obligation is a direct consequence of the transformations N and A_k (Lemma 2 and 3): We have $p'' \models \mathrm{Shift}_k(p,c) = A_2(m, N(e,c))$ iff $(p'' \circ m) \models N(e,c)$ iff $(p'' \circ m) \circ e \models c$ iff $(p'' \circ p) \models c$, as $(p'' \circ m)$ is in $\mathcal{M}$ and $(p'' \circ m) \circ e$ is an epi-$\mathcal{M}$-factorization of $(p'' \circ p)$. □

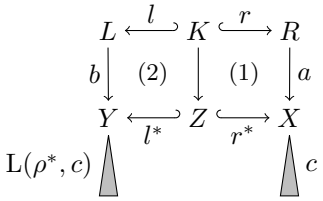
Second, there is a transformation from right to left application conditions such that a comatch satisfies an application condition iff the match satisfies the transformed application condition. This transformation is used in Section 7, e.g. in the construction of weakest preconditions. The construction is a generalized version of the corresponding construction for basic application conditions in (Ehrig et al. 2006b), first described in (Heckel and Wagner 1995).

Theorem 6 (transformation of application conditions). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object. There is a transformation L such that, for every rule ρ, every right application condition c for ρ, and every direct derivation $G \Rightarrow_{\rho,m,m^*} H$,

$$m \models L(\rho, c) \iff m^* \models c.$$

The idea of transformation L is to apply ρ and subsequent "derived" rules on the objects and morphisms of the condition, if possible, while preserving the (logical) structure of the condition.

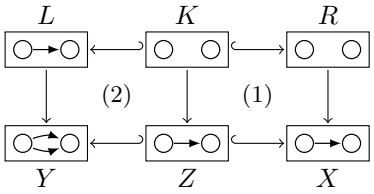
Construction. Transformation $L(\rho, c)$ is defined inductively as follows:



Let $L(\rho, \mathrm{true}) = \mathrm{true}$ and $L(\rho, \exists(a, c')) = \exists(b, L(\rho^*, c'))$ if $\langle r, a \rangle$ has a pushout complement (1) and $\rho^* = \langle Y \hookleftarrow Z \hookrightarrow X \rangle$ denotes the "derived" rule by constructing pushout (2). Note, as r is in $\mathcal{M}$, the pushout complement is unique, if existent. Otherwise, $\langle r, a \rangle$ has no pushout complement and $L(\rho, \exists(a, c')) = \mathrm{false}$.

For Boolean formulas over application conditions, the transformation L is extended in the usual way.

Example 15. The right application condition $\mathrm{ac} = \neg\exists(\bigcirc\ \bigcirc \to \bigcirc\!\!\rightarrow\!\!\bigcirc)$ for the rule $\rho = \langle \bigcirc\!\!\rightarrow\!\!\bigcirc \hookleftarrow \bigcirc\ \bigcirc \hookrightarrow \bigcirc\ \bigcirc \rangle$, meaning "An edge between the nodes in the comatch must not exist" is transformed into the left application condition $L(\rho, \mathrm{ac}) = \neg\exists(\bigcirc\!\!\rightarrow\!\!\bigcirc \to \bigcirc\rightleftarrows\bigcirc)$, meaning that two parallel edges between the nodes in the match must not exist.



Example 16. The right application condition $\mathrm{ac} = \neg\exists(\bigcirc\!\!\rightarrow\!\!\bigcirc \to \bigcirc\rightleftarrows\bigcirc)$, meaning "There do not exist two parallel edges between the nodes in the comatch", is transformed for the rule $\rho = \langle \bigcirc_1 \hookleftarrow \bigcirc_1 \hookrightarrow \bigcirc_1\!\!\rightarrow\!\!\bigcirc \rangle$ into the left application condition $L(\rho, \mathrm{ac}) = \neg\mathrm{false} \equiv$

true, meaning "The rule can always be applied", because the pair $\langle \bigcirc_1 \to \bigcirc_1 \to \bigcirc_2,\ \bigcirc_1 \to \bigcirc_2 \to \bigcirc_1 \rightleftarrows \bigcirc_2 \rangle$ of morphisms has no pushout complement.

Proof. By structural induction. Let $G \Rightarrow_{\rho,m,m^*} H$ be any direct derivation.
Basis. For $c = \text{true}$, we have $c = \text{true} = \mathrm{L}(\rho, \text{true}) = \mathrm{L}(\rho, c)$.
Hypothesis. Assume the statement holds for condition c'.
Step. For right application conditions of the form $c = \exists(a, c')$, we distinguish two cases:
Case 1. The pair $\langle r, a \rangle$ has a pushout complement. Then $\mathrm{L}(\rho, \exists(a, c')) = \exists(b, \mathrm{L}(\rho^*, c'))$ and we have: (A) Given a morphism $q' \colon Y \to G$ in $\mathcal{M}$ with $q' \circ b = m$, there is a decomposition of the pushouts of the derivation into pushouts (1')–(4') as follows: First, construct (4') as the pullback diagram of q' and d_2. By the universal property of pullbacks, there exists a unique morphism $K \to Z'$ such that the arising diagrams commute. By the pushout-pullback decomposition, (2') and (4') are pushouts. As $K \to Z'$ is unique, l is in $\mathcal{M}$, and pushout complements of morphisms in $\mathcal{M}$ are unique up to isomorphisms, pushout (2') equals pushout (2) from the construction up to isomorphism. Now construct pushout (1') of $K \to Z'$ and r. As $K \to Z'$ is unique, r is in $\mathcal{M}$, and pushout complements of morphisms in $\mathcal{M}$ are unique up to isomorphisms, pushout (1') equals pushout (1) from the construction up to isomorphism. By the universal property of pushouts, there is a unique morphism $q \colon X \to H$ with $q \circ a = m^*$. By the Decomposition Lemma of pushouts, the arising diagram (3') becomes a pushout. Since q' is in $\mathcal{M}$, the morphisms z and q are in $\mathcal{M}$, too.

$$
m \left(\begin{array}{ccccc}
L & \xleftarrow{l} & K & \xhookrightarrow{r} & R \\
{\scriptstyle b}\downarrow & (2') & \downarrow & (1') & \downarrow{\scriptstyle a} \\
Y & \xhookleftarrow{} & Z' & \xhookrightarrow{} & X \\
{\scriptstyle q'}\downarrow & (4) & \downarrow{\scriptstyle z} & (3) & \downarrow{\scriptstyle q} \\
G & \xleftarrow[d_2]{} & D & \xhookrightarrow[d_1]{} & H
\end{array} \right) m^*
$$

(B) Given a morphism $q \colon X \to H$ in $\mathcal{M}$ with $q \circ a = m^*$, the original pushouts can be decomposed into pushouts (1')–(4') similar as above: First (3') is constructed as pullback. Eventually a morphism $q' \colon Y \to G$ with $q' \circ b = m$ is yielded. Since q is in $\mathcal{M}$, z and q' are in $\mathcal{M}$.
(C) Given a right application condition c' and the derived rule $\rho^* = \langle Y \hookleftarrow Z \hookrightarrow X \rangle$ with morphisms $q' \colon Y \to G$ and $q \colon X \to H$, by the inductive hypothesis, we have $q \models c'$ iff $q' \models \mathrm{L}(\rho^*, c')$.

By the definitions of L and $\models$ and the statements above, we have

$$
\begin{array}{ll}
 & m \models \mathrm{L}(\rho, \exists(a, c')) = \exists(b, \mathrm{L}(\rho^*, c')) \\
\Leftrightarrow & \exists q' \colon Y \to G \text{ in } \mathcal{M}.\ q' \circ b = m \wedge q' \models \mathrm{L}(\rho^*, c') \\
\Leftrightarrow & \exists q \colon X \to H \text{ in } \mathcal{M}.\ q \circ a = m^* \wedge q \models c' \\
\Leftrightarrow & m^* \models \exists(a, c').
\end{array}
$$

Case 2. The pair $\langle r, a \rangle$ has no pushout complement and $\mathrm{L}(\rho, \exists(a, c')) = \text{false}$. We have to show that $m \models \text{false} \Leftrightarrow m^* \models \exists(a, c')$: As no morphism satisfies false, it suffices to show $m^* \not\models \exists(a, c')$. Assume $m^* \models \exists(a, c')$, then there exists some $q \colon X \to H$ with

$q \in \mathcal{M}$ and $q \circ a = m^*$. Thus there is a decomposition of the existing pushout into two pushouts (1) and (3) as in Case 1 above. Hence the pair $\langle r, a \rangle$ has a pushout complement. Contradiction.

For Boolean formulas over right application conditions, the statement follows directly from the definition and the inductive hypothesis. Thus, the statement holds for all right application conditions. □

In the case of $\mathcal{A}$-satisfiability, there is a derived transformation from right to left application conditions.

Corollary 4 ($\mathcal{A}$-satisfiability). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object, epi-$\mathcal{M}$-factorization and $\mathcal{M}$ strictly closed under decomposition. Then there is a derived transformation $\mathrm{L}_{\mathcal{A}}$ with $\mathrm{L}_{\mathcal{A}}(\rho, \mathrm{ac}) = \mathrm{Asat}(\mathrm{L}(\rho, \mathrm{Msat}(\mathrm{ac})))$ for $\mathcal{A}$-satisfiable conditions ac such that for every derivation $G \Rightarrow_{\rho,m,m^*} H$,

$$m \models_{\mathcal{A}} \mathrm{L}_{\mathcal{A}}(\rho, \mathrm{ac}) \iff m^* \models_{\mathcal{A}} \mathrm{ac}.$$

Proof. The statement follows immediately from Theorems 1, 2, and 6: $m \models_{\mathcal{A}} \mathrm{L}_{\mathcal{A}}(\rho, \mathrm{ac}) \Leftrightarrow m \models_{\mathcal{A}} \mathrm{Asat}(\mathrm{L}(\rho, \mathrm{Msat}(\mathrm{ac}))) \Leftrightarrow m \models \mathrm{L}(\rho, \mathrm{Msat}(\mathrm{ac})) \Leftrightarrow m^* \models \mathrm{Msat}(\mathrm{ac}) \Leftrightarrow m^* \models_{\mathcal{A}} \mathrm{ac}$. □

For weak adhesive HLR categories with $\mathcal{M}$-initial object, there are two transformations of application conditions to constraints, which correspond to the universal and existential closure of application conditions. In the case of $\mathcal{A}$-matching however, the closures have to be over all morphisms and do not fit to the notion of $\mathcal{M}$-satisfiability. Therefore the first part of the condition has to be transformed accordingly. Intuitively, these transformations construct weakest preconditions and strongest post conditions of the matching operation and are used in Section 7, e.g. in the construction of weakest preconditions and strongest postconditions.

Theorem 7 (transformation of application conditions into constraints). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object. There are transformations $\mathrm{C}_\forall$ and $\mathrm{C}_\exists$ such that, for every application condition ac over L and for all objects G,

$$\begin{aligned} G \models \mathrm{C}_\forall(\mathrm{ac}) &\quad\Leftrightarrow\quad \forall m\colon L \to G.\ m \models \mathrm{ac}, \quad \text{and} \\ G \models \mathrm{C}_\exists(\mathrm{ac}) &\quad\Leftrightarrow\quad \exists m\colon L \to G.\ m \models \mathrm{ac}. \end{aligned}$$

The idea of the transformations is to consider junctions of $\mathcal{M}$-satisfiable conditions, one condition for each possible level of identification.

Construction. For an application condition ac over L, let

$$\begin{aligned} \mathrm{C}_\forall(\mathrm{ac}) &= \wedge_{e \in \mathcal{E}}\, \forall(i_{L'},\ \mathrm{N}(e, \mathrm{ac})) \\ \mathrm{C}_\exists(\mathrm{ac}) &= \vee_{e \in \mathcal{E}}\, \exists(i_{L'},\ \mathrm{N}(e, \mathrm{ac})) \end{aligned}$$

where the set $\mathcal{E}$ ranges over all epimorphisms e starting from L, and, for an epimorphism $e\colon L \to L'$, $i_{L'}$ is the initial $\mathcal{M}$-morphism to L', and the transformation $\mathrm{N}(e, \_)$ is defined as in the construction of Lemma 2. For Boolean formulas over application conditions, the transformations $\mathrm{C}_\forall$ and $\mathrm{C}_\exists$ are extended in the usual way.

Proof. By Lemma 2, for all application conditions ac over L, and all morphisms m

with domain L with epi-$\mathcal{M}$-factorization $m = m' \circ e$, $m' \models \mathrm{N}(e, \mathrm{ac}) \Leftrightarrow m \models \mathrm{ac}$. Together with the definition of $\models$, $\wedge$, and $\mathrm{C}_\forall$, we have

$$\begin{aligned}
& \forall m\colon L \to G.\ m \models \mathrm{ac} \\
\Leftrightarrow\ & \forall e\colon L \to L' \text{ in } \mathcal{E}.\ \forall m'\colon L' \to G \text{ in } \mathcal{M}.\ m' \models \mathrm{N}(e, \mathrm{ac}) \\
\Leftrightarrow\ & \forall e\colon L \to L' \text{ in } \mathcal{E}.\ i_G \models \forall(i_{L'},\ \mathrm{N}(e, \mathrm{ac})) \\
\Leftrightarrow\ & \forall e\colon L \to L' \text{ in } \mathcal{E}.\ G \models \forall(i_{L'},\ \mathrm{N}(e, \mathrm{ac})) \\
\Leftrightarrow\ & G \models \wedge_{e \in \mathcal{E}}\, \forall(i_{L'},\ \mathrm{N}(e, \mathrm{ac})) \\
\Leftrightarrow\ & G \models \mathrm{C}_\forall(\mathrm{ac}).
\end{aligned}$$

The statement for $\mathrm{C}_\exists$ follows from the relationship $\mathrm{C}_\exists(\mathrm{ac}) \equiv \neg \mathrm{C}_\forall(\neg \mathrm{ac})$. □

Example 17. The application condition $\mathrm{ac} = \exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc)$ meaning "For the image of node 1, there exists some proper outgoing edge" is transformed into the constraints

$$\begin{aligned}
\mathrm{C}_\forall(\mathrm{ac}) &= \forall(\emptyset \to \bigcirc,\ \mathrm{N}(\mathrm{id}, \mathrm{ac})) = \forall(\emptyset \to \bigcirc,\ \mathrm{ac}) = \forall(\emptyset \to \bigcirc_1,\ \exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc)) \\
\mathrm{C}_\exists(\mathrm{ac}) &= \exists(\emptyset \to \bigcirc,\ \mathrm{N}(\mathrm{id}, \mathrm{ac})) = \exists(\emptyset \to \bigcirc,\ \mathrm{ac}) = \exists(\emptyset \to \bigcirc_1,\ \exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc))
\end{aligned}$$

meaning "For every node, there exists some proper outgoing edge" and "There exist a node with some proper outgoing edge", respectively.

Example 18. The application condition $\mathrm{ac} = \exists(\bigcirc\ \bigcirc \to \bigcirc)$ meaning "The two nodes have to be identified" is transformed into the constraints

$$\begin{aligned}
\mathrm{C}_\forall(\mathrm{ac}) &= \wedge_{e \in \mathcal{E}}\, \forall(e \circ i_L,\ \mathrm{N}(e, \mathrm{ac})) = \forall(\mathrm{id} \circ i_L,\ \mathrm{N}(\mathrm{id}, \mathrm{ac})) \wedge \forall(f \circ i_L,\ \mathrm{N}(f, \mathrm{ac})) \\
&= \forall(\emptyset \to \bigcirc\ \bigcirc,\ \exists(\bigcirc\ \bigcirc \to \bigcirc)) \wedge \forall(\emptyset \to \bigcirc,\ \exists(\bigcirc \to \bigcirc)) \\
&\equiv \forall(\emptyset \to \bigcirc\ \bigcirc,\ \mathrm{false}) \wedge \forall(\emptyset \to \bigcirc,\ \mathrm{true}) \\
&\equiv \neg\exists(\emptyset \to \bigcirc\ \bigcirc) \\
\mathrm{C}_\exists(\mathrm{ac}) &= \vee_{e \in \mathcal{E}}\, \exists(e \circ i_L,\ \mathrm{N}(e, \mathrm{ac})) = \exists(\mathrm{id} \circ i_L,\ \mathrm{N}(\mathrm{id}, \mathrm{ac})) \vee \exists(f \circ i_L,\ \mathrm{N}(f, \mathrm{ac})) \\
&= \exists(\emptyset \to \bigcirc\ \bigcirc,\ \exists(\bigcirc\ \bigcirc \to \bigcirc)) \vee \exists(\emptyset \to \bigcirc,\ \exists(\bigcirc \to \bigcirc)) \\
&\equiv \exists(\emptyset \to \bigcirc\ \bigcirc,\ \mathrm{false}) \vee \exists(\emptyset \to \bigcirc,\ \mathrm{true}) \\
&\equiv \exists(\emptyset \to \bigcirc)
\end{aligned}$$

where $i_L\colon \emptyset \to \bigcirc\ \bigcirc$ and $f\colon \bigcirc\ \bigcirc \to \bigcirc$, meaning "There exists at most one node" and "There exists a node", respectively.

If the matching and satisfiability notions correspond, i.e. in the case of $\mathcal{M}$-matching and $\mathcal{M}$-satisfiability and in the case of $\mathcal{A}$-matching and $\mathcal{A}$-satisfiability, $\mathrm{C}_\forall$ and $\mathrm{C}_\exists$ reduce to the universal and existential closure, respectively.

Remark. For $\mathcal{M}$-matching and $\mathcal{M}$-satisfiability and in the case of $\mathcal{A}$-matching and $\mathcal{A}$-satisfiability, one may simply define $\mathrm{C}_\forall(\mathrm{ac}) = \forall(i_L, \mathrm{ac})$ and $\mathrm{C}_\exists(\mathrm{ac}) = \exists(i_L, \mathrm{ac})$ using the universal and existential closure. In the first case, for every application condition ac over L and for all objects G,

$$\begin{aligned}
G \models \forall(i_L, \mathrm{ac}) &\Leftrightarrow \forall m\colon L \to G \in \mathcal{M}.\ m \models \mathrm{ac}, \\
G \models \exists(i_L, \mathrm{ac}) &\Leftrightarrow \exists m\colon L \to G \in \mathcal{M}.\ m \models \mathrm{ac}.
\end{aligned}$$

By definition of $\models$, we have: $G \models \forall(i_L, \mathrm{ac})$ iff $i_G \models \forall(i_L, \mathrm{ac})$ iff $\forall m\colon L \to G \in \mathcal{M}.\ m \models \mathrm{ac}$. The second case is analogous.

The applicability of a rule can be expressed by a left application condition on the matching morphism. We use this result in the construction of weakest preconditions in Section 7.

Theorem 8 (applicability of a rule). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$ strictly closed under decomposition. There is a transformation Def from rules to application conditions such that, for every rule ρ and every morphism $m\colon L \to G$,

$$m \models \mathrm{Def}(\rho) \Leftrightarrow \exists H.\ G \Rightarrow_{\rho,m,m^*} H.$$

In the graph case, the idea of the following construction is to express the gluing condition (Ehrig 1979) of a rule by a conjunction of negative application conditions.

Construction. For a rule $\rho = \langle q, \mathrm{ac} \rangle$, let $\mathrm{Def}(\rho) = \mathrm{Appl}(q) \wedge \mathrm{ac}_L \wedge \mathrm{L}(\rho, \mathrm{ac}_R)$, where, for a rule $q = \langle L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R \rangle$, $\mathrm{Appl}(q) = \wedge_{a \in A} \neg\exists a$ and the index set A ranges over all morphisms $a\colon L \to L'$ such that the pair $\langle l, a \rangle$ has no pushout complement and there is no decomposition $a = a'' \circ a'$ of morphism a such that a'' in $\mathcal{M}$ is not an isomorphism and $\langle l, a' \rangle$ has no pushout complement. The non-existence of such a decomposition ensures that we only consider minimal morphisms $a\colon L \to L'$. The morphism a'' is required to be non-isomorphic, otherwise there always is such a decomposition and A would be empty. The requirement a'' in $\mathcal{M}$ is necessary in context of an $\mathcal{M}$-satisfiable condition.

Remark. For the category $\langle \mathbf{Graphs}, \mathit{Inj} \rangle$, a morphism m satisfies the application condition $\mathrm{Appl}(\rho)$ iff m satisfies the contact and identification condition (Ehrig 1979). Note that the graph condition $\mathrm{Appl}(\rho)$ is finite, i.e. a finite conjunction of conditions (up to isomorphism).

Example 19. For $\texttt{DeleteNode} = \langle \bigcirc \hookleftarrow \emptyset \hookrightarrow \emptyset \rangle$, the application of `DeleteNode` requires the absence of additional edges adjacent to the deleted node:

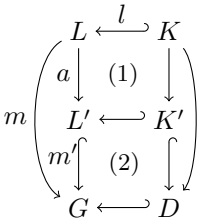
$$\mathrm{Appl}(\texttt{DeleteNode}) = \neg\exists(\bigcirc_1 \to \bigcirc_1 \to \bigcirc_2) \wedge \neg\exists(\bigcirc_1 \to \bigcirc_1 \leftarrow \bigcirc_2) \wedge \neg\exists(\bigcirc_1 \to \circlearrowright_1)$$

Proof. For plain rules q, we show that, for every morphism $m\colon L \to G$,

$$m \models \mathrm{Appl}(q) \Leftrightarrow \exists H. G \Rightarrow_{q,m,m^*} H.$$

Only if. Assume there is no direct derivation $G \Rightarrow_{q,m,m^*} H$, i.e. the pair $\langle l, m \rangle$ has no pushout complement. Let be $a = m$ and $m' = \mathrm{id}_G$. Now, you have some morphism a with $m = m' \circ a$ for some morphism m' in $\mathcal{M}$ such that $\langle l, a \rangle$ has no pushout complement. If there is a decomposition $a = a'' \circ a'$ of a such that $\langle l, a' \rangle$ has no pushout complement and a'' in $\mathcal{M}$ is not an isomorphism, let be $a = a'$, $m' = m' \circ a''$, m' in $\mathcal{M}$ and repeat the argument. Otherwise there is a no such decomposition and a belongs to the construction. As $m = m' \circ a$ and m' in $\mathcal{M}$, $m \models \exists a$ and $m \not\models \mathrm{Appl}(q)$.

If. Let $G \Rightarrow_{q,m,m^*} H$, i.e. the pair $\langle l, m\rangle$ has a pushout complement. Assume, there is $a \in A$ such that $m \models \exists a$, i.e. there is a morphism m' in $\mathcal{M}$ such that $m = m' \circ a$. Construct (2) as pullback of $L' \to G \leftarrow D$. By the universal property of pullbacks, there is a morphism $K \to K'$ such that the resulting diagrams commute. By the pushout-pullback decomposition, the pushout (1)+(2) has a decomposition into two pushouts (1) and (2) and, in particular, $\langle l, a\rangle$ has a pushout complement, contradiction. Consequently, for every morphism $a \in A$, $m \models \neg\exists a$ and $m \models \mathrm{Appl}(q)$.



By the definition of Def and $\models$, Theorem 8, the statement above, and the definition of $\Rightarrow$, for every morphism $m\colon L \to G$, $m \models \mathrm{Def}(\rho)$ iff $m \models \mathrm{Appl}(q) \wedge m \models \mathrm{ac}_L \wedge m \models \mathrm{L}(\rho, \mathrm{ac}_R)$ iff $\exists H.G \Rightarrow_{q,m,m^*} H \wedge m \models \mathrm{ac}_L \wedge m^* \models \mathrm{ac}_R$ iff $\exists H.G \Rightarrow_{\rho,m,m^*} H$. This completes the proof. □

7. Correctness of transformation systems

In this section we consider the correctness of transformation systems by integration of constraints into application conditions, and alternatively, by verification.

Our first result is that, given a specification in form of a constraint, any transformation rule can be "corrected" in the sense of preventing rule applications leading to objects not satisfying the constraint. More precisely, the transformations of Section 6 can be used to integrate constraints into left application conditions of a rule such that every direct derivation is constraint-guaranteeing or constraint-preserving.

Definition 9 (guarantee and preservation of constraints). Given a constraint c, a rule ρ is *c-guaranteeing* if for every direct derivation $G \Rightarrow_\rho H$, $H \models c$. A rule ρ is *c-preserving* if for every direct derivation $G \Rightarrow_\rho H$, $G \models c$ implies $H \models c$.

By Theorems 5 and 6, we obtain the following result. For a rule ρ, let $[\![\rho]\!]_{\mathcal{A}}$ denote the sets of pairs $\langle G, H\rangle$ with direct derivation $G \Rightarrow_{\rho,m,m^*} H$.

Corollary 5 (guarantee and preservation of constraints). Let $\langle \mathcal{C}, \mathcal{M}\rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object and epi-$\mathcal{M}$-factorization. There are transformations Gua, Pres from rules and conditions to rules such that for every transformation rule ρ, and every constraint c,

$$[\![\mathrm{Gua}(\rho, c)]\!]_{\mathcal{A}} \subseteq [\![\rho]\!]_{\mathcal{A}} \text{ and } \mathrm{Gua}(\rho, c) \text{ is } c\text{-guaranteeing, and}$$
$$[\![\mathrm{Pres}(\rho, c)]\!]_{\mathcal{A}} \subseteq [\![\rho]\!]_{\mathcal{A}} \text{ and } \mathrm{Pres}(\rho, c) \text{ is } c\text{-preserving.}$$

Construction. For a rule $\rho = \langle\langle L \hookleftarrow K \hookrightarrow R\rangle, \langle \mathrm{ac}_L, \mathrm{ac}_R\rangle\rangle$, let $\mathrm{Gua}(\rho, c) = \langle q, \langle \mathrm{ac}_L \wedge \mathrm{ac}_{gua}, \mathrm{ac}_R\rangle\rangle$ with $\mathrm{ac}_{gua} = \mathrm{L}(\rho, \mathrm{A}(\rho, c))$ and let $\mathrm{Pres}(\rho, c) = \langle q, \langle \mathrm{ac}_L \wedge \mathrm{ac}_{pres}, \mathrm{ac}_R\rangle\rangle$ with $\mathrm{ac}_{pres} = (\mathrm{A}(\rho^{-1}, c) \Rightarrow \mathrm{ac}_{gua})$, where ρ^{-1} denotes the inverse rule $\langle\langle R \hookleftarrow K \hookrightarrow L\rangle, \langle \mathrm{ac}_R, \mathrm{ac}_L\rangle\rangle$ of ρ.

Proof. Clearly, $[\![\mathrm{Gua}(\rho, c)]\!]_{\mathcal{A}} \subseteq [\![\rho]\!]_{\mathcal{A}}$ and $[\![\mathrm{Pres}(\rho, c)]\!]_{\mathcal{A}} \subseteq [\![\rho]\!]_{\mathcal{A}}$. Using Theorems 5 and 6, we show that $\mathrm{Gua}(\rho, c)$ is c-guaranteeing by showing that ac_{gua} guarantees the satisfiability of c: For every direct derivation $G \Rightarrow_{\mathrm{Gua}(\rho,c)} H$, $H \models c$ iff $m^* \models \mathrm{A}(\rho, c)$ iff

$m \models \mathrm{L}(\rho, \mathrm{A}(\rho, c))$ iff $m \models \mathrm{ac}_{gua}$. Finally, we show that $\mathrm{Pres}(\rho, c)$ is c-preserving by showing that ac_{pres} preserves the satisfiability of c: For every direct derivation $G \Rightarrow_{\mathrm{Pres}(\rho,c)} H$, ($G \models c$ implies $H \models c$) iff ($m \models \mathrm{A}(\rho^{-1}, c)$ implies $m \models \mathrm{ac}_{gua}$) iff $m \models (\mathrm{A}(\rho^{-1}, c) \Rightarrow \mathrm{ac}_{gua})$ iff $m \models \mathrm{ac}_{pres}$. □

The transformations Gua and Pres enforce correctness by restricting the applicability of transformation rules. In this way, a transformation system can be transformed into a constraint-guaranteeing or constraint-preserving system. Alternatively, one may use the transformations of Section 6 to verify the correctness of transformation systems with respect to given pre- and postconditions. A standard method is to construct a weakest precondition and to show that the precondition implies the weakest precondition.

In the following, we show how to construct weakest preconditions and strongest postconditions for transformation rules.

Definition 10 (weakest preconditions). A condition c is a *precondition* of a rule ρ relative to a condition d, if for all objects G with $G \models c$, $G \Rightarrow_\rho H$ implies $H \models d$ for all H, and $G \Rightarrow_\rho H$ for some H. A precondition c is a *weakest precondition* of ρ relative to d, denoted by $\mathrm{wp}(\rho, d)$, if any precondition of ρ relative to d implies c. Analogously, a condition c is a *liberal precondition*, if for all objects $G \models c$ at least $G \Rightarrow_\rho H$ implies $H \models d$ for all H and a *weakest liberal precondition*, denoted by $\mathrm{wlp}(\rho, d)$, if any liberal precondition of ρ relative to d implies c.

The following characterization points out a simple proof scheme for weakest liberal preconditions.

Fact 8 (characterization wlp**).** A condition c is a weakest liberal precondition of ρ relative to d if, for all objects G, ($G \models c$) iff ($G \Rightarrow_\rho H$ implies $H \models d$ for all H).

Proof. Assume, for all objects G, ($G \models c$) iff ($G \Rightarrow_\rho H$ implies $H \models d$ for all H). By Definition 10, c is a liberal precondition of ρ relative to d. It remains to show for any other liberal precondition c' of ρ relative to d, c' implies c. Let G be an arbitrary object and assume ($G \models c'$). According to Definition 10, we have ($G \Rightarrow_\rho H$ implies $H \models d$ for all H). Using the assumption, we have $G \models c$. As c' implies c, c is a weakest liberal precondition. □

For the construction of weakest preconditions, the following observation is essential:

Fact 9 (existence of results). $G \models \neg\mathrm{wlp}(\rho, \mathrm{false}) \Leftrightarrow G \Rightarrow_\rho H$ for some H.

Proof. There is an object H such that $G \Rightarrow_\rho H$, iff there is an object H such that $G \Rightarrow_\rho H$ and $H \models \mathrm{true}$, iff not for all objects H we have not ($G \Rightarrow_\rho H$ and $H \models \mathrm{true}$), iff not for all objects H we have not $G \Rightarrow_\rho H$ or $H \not\models \mathrm{true}$, iff not for all objects H we have $G \Rightarrow_\rho H$ implies $H \models \mathrm{false}$. □

Theorem 9 (weakest preconditions). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object, epi-$\mathcal{M}$-factorization and $\mathcal{M}$ strictly closed under decomposition. There are transformations Wlp and Wp such that for every rule ρ and every condition d,

$\mathrm{Wlp}(\rho, d)$ is a weakest liberal precondition and $\mathrm{Wp}(\rho, d)$ is a weakest precondition of ρ relative to d.

Construction. For a rule ρ and condition d, let $\mathrm{Wlp}(\rho, d) = \mathrm{C}_\forall(\mathrm{Def}(\rho) \Rightarrow \mathrm{L}(\rho, \mathrm{A}(\rho, d)))$, and $\mathrm{Wp}(\rho, d) = \mathrm{Wlp}(\rho, d) \wedge \neg\mathrm{Wlp}(\rho, \mathrm{false})$.

Proof. We show $\mathrm{Wlp}(\rho, d) \equiv \mathrm{wlp}(\rho, d)$ for arbitrary rules ρ and conditions d: For all objects G, we have:

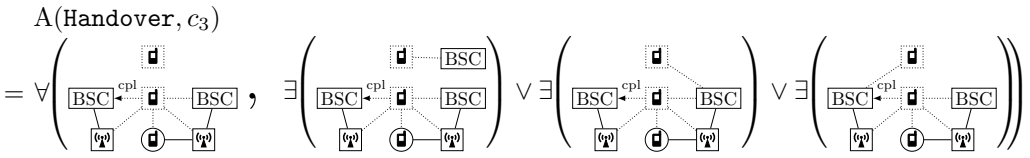
$$
\begin{array}{lll}
 & G \models \mathrm{Wlp}(\rho, d) & \\
\Leftrightarrow & G \models \mathrm{C}_\forall(\mathrm{Def}(\rho) \Rightarrow \mathrm{L}(\rho, \mathrm{A}(\rho, d))) & \text{(Def. Wlp)} \\
\Leftrightarrow & \forall L \overset{m}{\to} G.\ m \models (\mathrm{Def}(\rho) \Rightarrow \mathrm{L}(\rho, \mathrm{A}(\rho, d))) & \text{(Thm. 7)} \\
\Leftrightarrow & \forall L \overset{m}{\to} G.\ m \models \mathrm{Def}(\rho) \Rightarrow m \models \mathrm{L}(\rho, \mathrm{A}(\rho, d)) & \text{(Def. } \models\text{)} \\
\Leftrightarrow & \forall L \overset{m}{\to} G, R \overset{m^*}{\to} H.\ m \models \mathrm{Def}(\rho) \Rightarrow m^* \models \mathrm{A}(\rho, d) & \text{(Thm. 6)} \\
\Leftrightarrow & \forall L \overset{m}{\to} G, R \overset{m^*}{\to} H.\ (G \Rightarrow_{\rho,m,m^*} H) \Rightarrow H \models d & \text{(Thms. 8 \& 5)} \\
\Leftrightarrow & G \models \mathrm{wlp}(\rho, d). & \text{(Def. wlp, Fact 8)}
\end{array}
$$

Consequently, $\mathrm{Wlp}(\rho, d)$ is a weakest liberal precondition of ρ relative to d. By Fact 9, Wp is a weakest precondition of ρ relative to d. □

Example 20 (GSM weakest preconditions). Consider the GSM condition *consistency* in Example 4 and the GSM transformation system in Example 7. Weakest preconditions can be used to answer the question, whether or not the GSM transformation system is correct with respect to the pre- and postcondition *consistency*. We focus on the rule `Handover` and the subcondition c_3 of *consistency*.

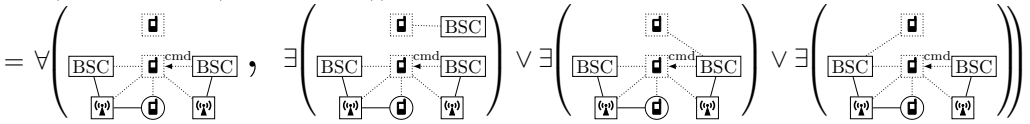
The application of the transformation A to the rule `Handover` and the constraint c_3 yields a right application condition, i.e. a condition over the right-hand side of `Handover`:

$\mathrm{A}(\texttt{Handover}, c_3)$



The application of the transformation L to the rule `Handover` and the right application condition $\mathrm{A}(\texttt{Handover}, c_3)$ yields a left application condition, i.e. a condition over the left-hand side of `Handover`:

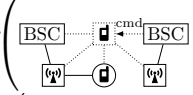
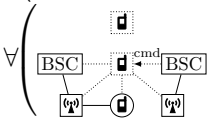
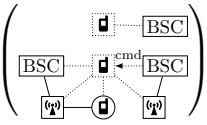
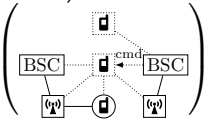
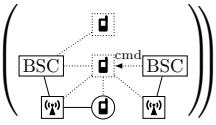
$\mathrm{L}(\texttt{Handover},\ \mathrm{A}(\texttt{Handover}, c_3))$



The application of the transformation Appl to the rule `Handover` yields the left application condition $\mathrm{Appl}(\texttt{Handover}) = \mathrm{true}$ because only injective matches are considered and the rule `Handover` is "node-preserving", thus, for every match, there exists a pushout complement.

Finally, the application of the transformation Wlp to the rule `Handover` and the con-

dition c_3 yields the following constraint over the empty graph:

$$
\begin{aligned}
&\mathrm{Wlp}(\texttt{Handover}, c_3)\\
&= \mathrm{C}_\forall(\mathrm{Def}(\texttt{Handover}) \Rightarrow \mathrm{L}(\texttt{Handover}, \mathrm{A}(\texttt{Handover}, c_3)))\\
&= \mathrm{C}_\forall((\mathrm{Appl}(\texttt{Handover}) \wedge \mathrm{true} \wedge \mathrm{true}) \Rightarrow \mathrm{L}(\texttt{Handover}, \mathrm{A}(\texttt{Handover}, c_3)))\\
&\equiv \mathrm{C}_\forall(\mathrm{true} \Rightarrow \mathrm{L}(\texttt{Handover}, \mathrm{A}(\texttt{Handover}, c_3)))\\
&\equiv \mathrm{C}_\forall(\mathrm{L}(\texttt{Handover}, \mathrm{A}(\texttt{Handover}, c_3)))\\
&= \forall\Big(\text{  }\text{}\ ,\ \mathrm{L}(\texttt{Handover}, \mathrm{A}(\texttt{Handover}, c_3))\Big)\\
&\equiv \forall\Big(\text{  }\text{}\ ,\ \exists\Big(\text{  }\text{}\Big) \vee \exists\Big(\text{  }\text{}\Big) \vee \exists\Big(\text{  }\text{}\Big)\Big)
\end{aligned}
$$

meaning "Every mobile station record outside the scope of the rule `Handover` has to be associated with a base transceiver station (outside or inside the occurrence of the left-hand side)". Obviously, the constraint c_3 implies $\mathrm{Wlp}(\texttt{Handover}, c_3)$.

The rule `Handover` is correct with respect to the pre- and postcondition *consistency*, provided that the graph condition *consistency* implies $\mathrm{Wlp}(\texttt{Handover}, \textit{consistency})$. The GSM transformation system is correct, provided that all rules are correct.

Complementary to weakest liberal preconditions, one may consider strongest postconditions, i.e. the strongest condition one can assume to hold in every state after the application of a rule onto an object satisfying a given condition.

Definition 11 (strongest postconditions). A condition c is a *strongest postcondition* of a rule ρ relative to a condition d, denoted by $\mathrm{sp}(\rho, d)$, if for all objects H, H satisfies c iff there is some object G such that $G \Rightarrow_\rho H$ and $G \models d$.

As a rule is applicable to G if and only if the inverse rule is applicable to H, we have the following relationship between weakest liberal preconditions and strongest postconditions:

Fact 10 (relationship of sp **and** wlp**).** $\mathrm{sp}(\rho, d) \equiv \neg\mathrm{wlp}(\rho^{-1}, \neg d)$.

Proof. This relationship may be seen as follows:

$$
\begin{aligned}
& H \models \neg\mathrm{wlp}(\rho^{-1}, \neg d) && \\
\Leftrightarrow\ & \neg\forall G.\ (H \Rightarrow_{\rho^{-1}} G) \Rightarrow G \models \neg d && \text{(Def. wlp)}\\
\Leftrightarrow\ & \neg\forall G.\ (G \Rightarrow_{\rho} H) \Rightarrow G \models \neg d && \text{(Def. } \rho^{-1}\text{)}\\
\Leftrightarrow\ & \exists G.\ (G \Rightarrow_{\rho} H) \wedge G \models d && \text{(Def. } \forall, \Rightarrow, \vee\text{)}\\
\Leftrightarrow\ & H \models \mathrm{sp}(\rho, d) && \text{(Def. sp)}
\end{aligned}
$$

Consequently, the mentioned relationship of sp and wlp holds. □

As an immediate consequence of Fact 10, we can express the construction of Sp in terms of basic transformations.

Corollary 6 (strongest postconditions). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with $\mathcal{M}$-initial object, epi-$\mathcal{M}$-factorization and $\mathcal{M}$ strictly closed under decomposition. There is a transformation Sp such that for every rule ρ and every condition d, $\mathrm{Sp}(\rho, d)$ is a strongest postcondition of ρ relative to d.

Construction. For arbitrary rules ρ and conditions d, we have

$$\mathrm{Sp}(\rho, d) = \mathrm{C}_\exists(\mathrm{Def}(\rho^{-1}) \wedge \mathrm{L}(\rho^{-1}, \mathrm{A}(\rho^{-1}, d))).$$

Proof. Let ρ be an arbitrary rule and d be an arbitrary condition.

$$\begin{aligned}
&\mathrm{Sp}(\rho, d) \\
\equiv\ &\mathrm{C}_\exists(\mathrm{Def}(\rho^{-1}) \wedge \mathrm{L}(\rho^{-1}, \mathrm{A}(\rho^{-1}, d))) && \text{(Def. Sp)} \\
\equiv\ &\neg\mathrm{C}_\forall(\mathrm{Def}(\rho^{-1}) \Rightarrow \neg\mathrm{L}(\rho^{-1}, \mathrm{A}(\rho^{-1}, d))) && \text{(Def. } \mathrm{C}_\exists, \mathrm{C}_\forall, \Rightarrow, \vee) \\
\equiv\ &\neg\mathrm{C}_\forall(\mathrm{Def}(\rho^{-1}) \Rightarrow \mathrm{L}(\rho^{-1}, \mathrm{A}(\rho^{-1}, \neg d))) && \text{(Def. A, L)} \\
\equiv\ &\neg\mathrm{Wlp}(\rho^{-1}, \neg d) && \text{(Def.,Correct. Wlp)} \\
\equiv\ &\mathrm{sp}(\rho, d) && \text{(Fact 10)}
\end{aligned}$$

Consequently, Sp is a strongest postcondition of ρ relative to d. □

The implication problem of conditions, seen as a tautology or satisfiability problem, is covered in (Orejas et al. 2008; Orejas 2008; Pennemann 2008a; Pennemann 2008b), although the first two papers consider conditions as in (Ehrig et al. 2006b). Alternatively, in the next section, we consider a translation of graph conditions into first-order graph formulas, making it possible to apply existing first-order theorem provers and satisfiability solvers.

8. Expressiveness of graph conditions

We are interested in classifying nested conditions, i.e. we want to classify the kind of graph properties that can be expressed by graph conditions, and we want to know if it is decidable, whether or not a graph condition is satisfiable at all (satisfiability problem), or whether or not a graph condition is always valid (tautology problem). To this effect, we compare graph conditions and first-order formulas on graphs (Courcelle 1990; Courcelle 1997) and show that the concepts are expressively equivalent. More precisely, we show that there are transformations from $\mathcal{A}$-satisfiable graph conditions into equivalent graph formulas and vice versa, similar to (Rensink 2004). However, we consider graphs with parallel edges, i.e. multiple, distinguishable edges which may have the same label. Together with the transformations between $\mathcal{M}$- and $\mathcal{A}$-satisfiability, we yield the wanted result. As an additional result, the transformations enable the use of existing first-order tools to tackle the tautology and satisfiability problem of graph conditions.

For simplicity, we consider graphs with a common labeling function for nodes and edges, while maintaining the disjointness of the node and edge alphabet. Note that all considerations can be done for graphs with separate labeling functions, as well. The definition of first-order graph formulas is similar to (Courcelle 1990; Courcelle 1997): we allow quantification over nodes and edges, consider a tertiary incidence relation and introduce a unary predicate for each label. For a fixed label alphabet $\mathrm{C} = \langle \mathrm{C_V}, \mathrm{C_E} \rangle$ with

$C_V \cap C_E = \emptyset$, the induced signature $\Sigma = (\emptyset, \{\mathrm{lab}_b \mid b \in C\} \cup \{\mathrm{inc}, =\})$ contains a unary predicate symbol lab_b for every label b, a tertiary predicate symbol inc and a binary predicate symbol $=$.

Definition 12 (first-order graph formulas). Let Var be an infinite, countable set of variables. The set of all *(first-order graph) formulas* over Σ is inductively defined: For $b \in C$ and $x, y, z \in \mathrm{Var}$, $\mathrm{lab}_b(x)$, $\mathrm{inc}(x, y, z)$ and $x = y$ are formulas over Σ. For formulas F, F_j $(j \in J)$ over Σ and $x \in \mathrm{Var}$, true, $\neg F$, $\wedge_{j \in J} F_j$, and $\exists x\ F$ are formulas over Σ. Additionally, false abbreviates $\neg\mathrm{true}$, $\vee_{j \in J} F_j$ abbreviates $\neg \wedge_{j \in J} \neg F_j$, $F \Rightarrow G$ abbreviates $\neg F \vee G$, $\forall x\ F$ abbreviates $\neg\exists x\ \neg F$, $\mathrm{edge}(x)$ abbreviates $\exists y\ \exists z\ \mathrm{inc}(x, y, z)$, and $\mathrm{node}(x)$ abbreviates $\neg\mathrm{edge}(x)$. For a formula F, $\mathrm{Free}(F)$ denotes the set of all *free* variables of F. A formula is *closed*, if $\mathrm{Free}(F) = \emptyset$, i.e. if F does not contain free variables.

The semantics of graph formulas are given in terms of a domain of values and an interpretation of the (non-logical) symbols of Σ.

Definition 13 (semantics of graph formulas). For a non-empty graph G, let (D_G, I_G) be the induced Σ-*structure* consisting of a non-empty domain $D_G = V_G + E_G$ and the interpretation I_G of the predicate symbols with $I_G(\mathrm{lab}_b)(d) = \mathit{true}$ iff $l_G(d) = b$, $I_G(\mathrm{inc})(e, u, v) = \mathit{true}$ iff $e \in E_G$, $s_G(e) = u$ and $t_G(e) = v$, and $I_G(=)(d, d') = \mathit{true}$ iff $d = d'$. The *semantic* $G[\![F]\!](\sigma)$ of a formula F over Σ in the graph G under the assignment $\sigma\colon \mathrm{Var} \to D_G$ is inductively defined by:

$G[\![p(x_1, \ldots, x_n)]\!](\sigma) = I_G(p)(\sigma(x_1), \ldots, \sigma(x_n))$ for an n-ary predicate p.

$G[\![\exists x\ F]\!](\sigma) = \mathit{true}$ iff there exists $d \in D_G$ such that $G[\![F]\!](\sigma\{x/d\}) = \mathit{true}$, where $\sigma\{x/d\}$ is the modified assignment with $\sigma\{x/d\}(x) = d$ and $\sigma\{x/d\}(y) = \sigma(y)$ otherwise.

The semantics is extended to the operators true, $\neg$ and $\wedge$ in the usual way.

A graph G *satisfies* a formula F, denoted by $G \models F$, iff for all assignments $\sigma\colon \mathrm{Var} \to D_G$, $G[\![F]\!](\sigma) = \mathit{true}$.

Example 21. The first-order graph formula

$$F = \mathrm{node}(u) \wedge \mathrm{lab}_0(u) \Rightarrow \exists v \exists e\ \mathrm{node}(v) \wedge \mathrm{lab}_1(v) \wedge \mathrm{inc}(e, u, v) \wedge \mathrm{lab}_a(e).$$

has the meaning "Whenever there is a node with label 0, then there exists an edge with label a from this node to a node with label 1".

For automated theorem proving, one requires an exact axiomatization of the above structures to restrict considerations to directed, totally labeled graphs, see (Courcelle 1997) for unlabeled graphs.

Fact 11 (axiomatization). For an alphabet $C = \langle C_V, C_E \rangle$ with $C_V \cap C_E = \emptyset$, the class of structures (D_G, I_G) over Σ are exactly those which satisfy the following properties:

(1) $\forall e\ \forall x\ \forall y\ (\mathrm{inc}(e, x, y) \Rightarrow \neg\exists u\ \exists v\ (\mathrm{inc}(x, u, v) \vee \mathrm{inc}(y, u, v)))$

(2) $\forall e\ \forall x\ \forall y\ \forall x'\ \forall y'\ ((\mathrm{inc}(e, x, y) \wedge \mathrm{inc}(e, x', y')) \Rightarrow (x = x' \wedge y = y'))$

(3) $\forall x\ \wedge_{b,d \in C_V, b \neq d} \neg(\mathrm{lab}_b(x) \wedge \mathrm{lab}_d(x))$ and $\forall x\ \wedge_{b,d \in C_E, b \neq d} \neg(\mathrm{lab}_b(x) \wedge \mathrm{lab}_d(x))$

(4) $\forall x\ \mathrm{node}(x) \Leftrightarrow \vee_{b \in C_V}\ \mathrm{lab}_b(x)$ and $\forall x\ \mathrm{edge}(x) \Leftrightarrow \vee_{b \in C_E}\ \mathrm{lab}_b(x)$

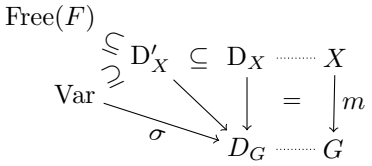
The meaning of the axioms is: (1) (target and source) nodes cannot be edges, (2) an edge has at most one source and one target, (3) an element has at most one label, and (4) every node has a node label, every edge has an edge label. Every other statement is implicit, such as "An edge has source and target nodes" (otherwise, it is not an edge).

There is a transformation from first-order graph formulas into graph conditions.

Theorem 10 (from formulas to conditions). There is a transformation Cond from first-order graph formulas to graph conditions, such that, for all first-order formulas F over Σ and all graphs G,

$$G \models F \iff G \models_{\mathcal{A}} \mathrm{Cond}(F).$$

Before we give a construction of the transformation Cond, let us make some preliminary considerations. We consider formulas on directed, labeled graphs with multiple parallel edges. Hence edges are handled as individuals. If F is a *rectified* formula, i.e. distinct quantifiers bind occurrences of distinct variables, the variables of F can be represented by isolated nodes and edges in the graphs of a constructed condition. Let X be such a graph. If the set $\mathrm{D}'_X = \mathrm{Iso}_X + \mathrm{E}_X$ of all isolated nodes and all edges in X is a subset of the set Var of variables, then every graph morphism $m\colon X \to G$ into a non-empty graph G induces an assignment $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$ such that $m = \sigma[\mathrm{D}'_X]$, i.e. $m(x) = \sigma(x)$ for each $x \in \mathrm{D}'_X$. Vice versa, an assignment $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$ induces a mapping $\mathrm{D}'_X \to \mathrm{D}_G$ that may be extended to a graph morphism $m\colon X \to G$ with $m = \sigma[\mathrm{D}'_X]$.



The key idea of the transformation Cond is to represent existential quantification in a formula by disjunction over all possible choices, i.e. nodes or edges with all their possible labels. Some of these branches may later become unsatisfiable, depending on occurring lab predicates.

Construction. Assume that F is a closed and rectified formula over Σ. Otherwise, consider the universal closure of F and rename the variables. The graph condition is given by $\mathrm{Cond}(F) = \mathrm{Cond}(\emptyset, F)$, where $\emptyset$ denotes the empty graph. For a formula F over Σ and a graph X with $\mathrm{Free}(F) \subseteq \mathrm{D}'_X \subseteq \mathrm{Var}$, the graph condition $\mathrm{Cond}(X, F)$ is constructed as follows:

$\mathrm{Cond}(X, \mathrm{lab}_b(x)) =$ true if $\mathrm{l}_X(x) = b$; false otherwise.

$\mathrm{Cond}(X, \mathrm{inc}(x, y, z)) = \exists(X \to X[\mathrm{s}_X(x) = y]) \wedge \exists(X \to X[\mathrm{t}_X(x) = z])$ if $x \in \mathrm{E}_X$ and $y, z \in \mathrm{V}_X$; false otherwise.

A graph $X[x = y]$ is obtained from X by identifying the elements x and y.

$\mathrm{Cond}(X, x = y) = \exists(X \to X[x = y])$ iff x and y are identifiable, i.e. ($x, y \in \mathrm{V}_X$ or $x, y \in \mathrm{E}_X$, $\mathrm{s}_X(x) = \mathrm{s}_X(y)$, $\mathrm{t}_X(x) = \mathrm{t}_X(y)$) and $\mathrm{l}_X(x) = \mathrm{l}_X(y)$; false otherwise.

$\mathrm{Cond}(X, \exists x\, F) = \vee_{b \in \mathrm{C_V}} \exists(X \to Y, \mathrm{Cond}(Y, F)) \vee \vee_{b \in \mathrm{C_E}, d, d' \in \mathrm{C_V}} \exists(X \to Z, \mathrm{Cond}(Z, F))$

where $Y = X + \overset{x}{\textcircled{b}}$ is obtained from X by adding a node x with label b and $Z = X + \overset{}{\textcircled{d}} \xrightarrow[x]{b} \overset{}{\textcircled{d'}}$ by adding an edge x with label b together with a d-labeled source and a d'-labeled target.

$\mathrm{Cond}(X, F)$ is extended to Boolean formulas with the operators true, $\neg$, $\wedge$ as usual.

Example 22. Let $\mathrm{C_V} = \{0, 1\}$ and $\mathrm{C_E} = \{a, b\}$. The first-order graph formula $F = \exists x\ \mathrm{lab}_a(x)$ with the meaning "There exists an item with label a" is transformed into the graph condition

$$
\begin{aligned}
& \mathrm{Cond}(\exists x\ \mathrm{lab}_a(x)) = \mathrm{Cond}(\emptyset, \exists x\ \mathrm{lab}_a(x)) \\
= \quad & \vee_{m \in \mathrm{C_V}} \exists(\emptyset \to \overset{x}{\textcircled{m}},\ \mathrm{Cond}(\overset{x}{\textcircled{m}}, \mathrm{lab}_a(x))) \\
& \vee \vee_{k,n \in \mathrm{C_V}, m \in \mathrm{C_E}} \exists(\emptyset \to \overset{}{\textcircled{k}} \xrightarrow[x]{m} \overset{}{\textcircled{n}},\ \mathrm{Cond}(\overset{}{\textcircled{k}} \xrightarrow[x]{m} \overset{}{\textcircled{n}}, \mathrm{lab}_a(x))) \\
= \quad & \exists(\emptyset \to \overset{}{\textcircled{0}},\ \mathrm{false}) \vee \exists(\emptyset \to \overset{}{\textcircled{1}},\ \mathrm{false}) \\
& \vee \vee_{k,n \in \mathrm{C_V}} (\exists(\emptyset \to \overset{}{\textcircled{k}} \xrightarrow[x]{a} \overset{}{\textcircled{n}},\ \mathrm{true}) \vee \exists(\emptyset \to \overset{}{\textcircled{k}} \xrightarrow[x]{b} \overset{}{\textcircled{n}},\ \mathrm{false})) \\
\equiv \quad & \vee_{k,n \in \mathrm{C_V}} \exists(\emptyset \to \overset{}{\textcircled{k}} \xrightarrow[x]{a} \overset{}{\textcircled{n}}),
\end{aligned}
$$

with the meaning "There exists an edge with label a and arbitrarily labeled source and target". This example shows that, in case of totally labeled graphs, unspecifiedness is represented by disjunction over all possibilities, which may, at least temporarily, lead to rather large conditions. A remedy could be the consideration of conditions over partial and/or partially labeled graphs.

The proof of Theorem 10 depends on the following lemma.

Lemma 4. For all rectified formulas F over Σ, all graphs G, and all graphs X with $\mathrm{Free}(F) \subseteq \mathrm{D}'_X \subseteq \mathrm{Var}$ we have: For all morphisms $m\colon X \to G$ and all assignments $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$ with $m = \sigma[\mathrm{D}'_X]$, $G[\![F]\!](\sigma) = \mathit{true}$ iff $m \models_{\mathcal{A}} \mathrm{Cond}(X, F)$.

Proof. By structural induction.

Basis. For $F = \mathrm{true}$, the statement is straightforward. For atomic formulas, the statement follows directly from the definitions:

(1) $G[\![\mathrm{lab}_b(x)]\!](\sigma) = \mathit{true}$
$\Leftrightarrow$ $\mathrm{l}_G(\sigma(x)) = b$ (Definition of $[\![\,]\!]$)
$\Leftrightarrow$ $\mathrm{l}_G(m(x)) = b$ ($m = \sigma[\mathrm{D}'_X]$, $*$)
$\Leftrightarrow$ $m \models_{\mathcal{A}} \mathrm{true}$ and $\mathrm{l}_X(x) = b$ (m label-preserving)
$\Leftrightarrow$ $m \models_{\mathcal{A}} \mathrm{Cond}(X, \mathrm{lab}_b(x))$ (Definition of Cond)
* $x \in \mathrm{Free}(\mathrm{lab}_b(x)) \subseteq \mathrm{D}'_X$

(2) $G[\![\mathrm{inc}(x, y, z)]\!](\sigma) = \mathit{true}$
$\Leftrightarrow$ $\sigma(x) \in \mathrm{E}_G,\ \mathrm{s}_G(\sigma(x)) = \sigma(y)$ and $\mathrm{t}_G(\sigma(x)) = \sigma(z)$ (Definition of $[\![\,]\!]$)
$\Leftrightarrow$ $\mathrm{m}(x) \in \mathrm{E}_G,\ \mathrm{s}_G(m(x)) = m(y)$ and $\mathrm{t}_G(m(x)) = m(z)$ ($m = \sigma[\mathrm{D}'_X]$, $*$)
$\Leftrightarrow$ $x \in \mathrm{E}_X,\ y, z \in \mathrm{V}_X,\ m(\mathrm{s}_X(x)) = m(y)$ and $m(\mathrm{t}_X(x)) = m(z)$ (m is a morphism)
$\Leftrightarrow$ $m \models_{\mathcal{A}} \exists(X \to X[\mathrm{s}_X(x) = y]) \wedge \exists(X \to X[\mathrm{t}_X(x) = y])$
and $x \in \mathrm{E}_X,\ y, z \in \mathrm{V}_X$ (Definition of $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $m \models_{\mathcal{A}} \mathrm{Cond}(X, \mathrm{inc}(x, y, z))$ (Definition of Cond)
* $x, y, z \in \mathrm{Free}(\mathrm{inc}(x, y, z)) \subseteq \mathrm{D}'_X$

(3) $G[\![x=y]\!](\sigma) = \mathit{true}$
$\Leftrightarrow$ $(\sigma(x),\sigma(y) \in \mathrm{V}_G$ or $\sigma(x),\sigma(y) \in \mathrm{E}_G)$ and $\sigma(x)=\sigma(y)$ (Definition of $[\![]\!]$)
$\Leftrightarrow$ $m(x)=m(y)$ $(m=\sigma[\mathrm{D}'_X], *)$
$\Leftrightarrow$ $m \models_{\mathcal{A}} \exists(X \to X[x=y])$ and x,y are identifiable (Definition of $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $m \models_{\mathcal{A}} \mathrm{Cond}(X, x=y)$ (Definition of Cond)
* $x,y \in \mathrm{Free}(x=y) \subseteq \mathrm{D}'_X$, m is a morphism

Hypothesis. Assume, the statement holds for rectified formulas F and F_j $(j \in J)$.
Step. For formulas of the form $\exists x\ F$, the proof uses the inductive hypothesis:

(4) $G[\![\exists x\ F]\!](\sigma) = \mathit{true}$
$\Leftrightarrow$ $\exists o \in \mathrm{D}_G.\ G[\![F]\!](\sigma\{x/o\}) = \mathit{true}$ (Def. $[\![]\!]$)
$\Leftrightarrow$ $\exists o \in \mathrm{V}_G.\ \exists b \in \mathrm{C}_\mathrm{V}.\ \mathrm{l}_G(o) = b$ and $G[\![F]\!](\sigma\{x/o\}) = \mathit{true}$ or
$\exists o \in \mathrm{E}_G.\ \exists b \in \mathrm{C}_\mathrm{E}.\ \mathrm{l}_G(o) = b$ and $G[\![F]\!](\sigma\{x/o\}) = \mathit{true}$ (Assignment)
$\Leftrightarrow$ $\exists b \in \mathrm{C}_\mathrm{V}.\ \exists m'.\ m = m' \circ X \to Y$ and $m' \models_{\mathcal{A}} \mathrm{Cond}(Y,F)$ or
$\exists b \in \mathrm{C}_\mathrm{E}.\ \exists d,d' \in \mathrm{C}_\mathrm{V}.\ \exists m'.\ m = m' \circ X \to Z,\ m' \models_{\mathcal{A}} \mathrm{Cond}(Z,F)$ (Hypothesis, $*$)
$\Leftrightarrow$ $\exists b \in \mathrm{C}_\mathrm{V}.\ m \models_{\mathcal{A}} \exists(X \to Y,\ \mathrm{Cond}(Y,F))$ or
$\exists b \in \mathrm{C}_\mathrm{E}.\ \exists d,d' \in \mathrm{C}_\mathrm{V}.\ m \models_{\mathcal{A}} \exists(X \to Z,\ \mathrm{Cond}(Z,F))$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $m \models_{\mathcal{A}} \mathrm{Cond}(X, \exists x\ F)$ (Def. Cond)
* $m' = \sigma\{x/o\}[\mathrm{D}'_X]$

where $Y = X + \overset{b}{\underset{x}{\textcircled{b}}}$ is obtained from X by adding a node x with label b and $Z = X + \textcircled{d} \xrightarrow[x]{b} \textcircled{d'}$ by adding an edge x with label b together with d-labeled source and d'-labeled target nodes. The rectifiedness of F guarantees that x does not already exists. For formulas built with the operators $\neg$ and $\wedge$, the proof of the statement is straightforward and uses the inductive hypothesis. This completes the inductive proof. □

Proof of Theorem 10. For all closed, rectified formulas F over Σ and all graphs G, we have the following: $G \models F$ iff for all assignments $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$, $G[\![F]\!](\sigma) = \mathit{true}$ (Definition $\models$) iff for all morphisms $i_G\colon \emptyset \to G$, $i_G \models_{\mathcal{A}} \mathrm{Cond}(\emptyset, F)$ (Lemma 4, for $\mathrm{Free}(F) = \emptyset = \mathrm{D}'_X$ and $X = \emptyset$) iff $G \models_{\mathcal{A}} \mathrm{Cond}(\emptyset, F)$ (Definition $\models_{\mathcal{A}}$). □

Vice versa, there is a transformation from graph conditions into first-order graph formulas.

Theorem 11 (from conditions to formulas). There exists a transformation Form from graph conditions to first-order graph formulas, such that, for all graph conditions c over the empty graph $\emptyset$ and all graphs G,

$$G \models_{\mathcal{A}} c \iff G \models \mathrm{Form}(c).$$

For the construction of the transformation Form, we consider morphisms and conditions in a certain normal form, such that the identities of the nodes and edges in a condition can be associated with variables, i.e., no unnecessary identity changes of elements take place within a condition. A morphism $a\colon P \to C$ is said to be *identity-preserving*, if $y \in C$ implies $(y = a(y)$ or $(y \notin a(P)$ and $y \notin P))$, elementwise for nodes and edges. This is no restriction, as every condition may be transformed into an equivalent condition, by subsequently replacing all morphisms not in this form by morphisms with this property.

Fact 12. Every identity-preserving graph morphism a with domain P is either the identity $\mathrm{id}\colon P \to P$ or can be decomposed into morphisms that, starting from P, subsequently add nodes, add edges, identify nodes, and identify edges. There may be multiple decompositions for a given morphism.

The key idea of the transformation Form is to decompose the condition $\exists(a, c')$ into a nested condition $\exists(a_1, \ldots, \exists(a_n, \exists(\mathrm{id}, c'))\ )$, such that in each morphism a_j, $1 \leq j \leq n$, one element is added or two elements are identified.

Construction. For conditions based on identity-preserving graph morphisms, Form is defined as follows:

$$\begin{aligned}
&\mathrm{Form}(\mathrm{true}) && = \mathrm{true}\\
&\mathrm{Form}(\exists(\mathrm{id}, c')) && = \mathrm{Form}(c')\\
&\mathrm{Form}(\exists([vb]a, c')) && = \exists v\ (\mathrm{node}(v) \wedge \mathrm{lab}_b(v) \wedge \mathrm{Form}(\exists(a, c')))\\
&\mathrm{Form}(\exists([euvb]a, c')) && = \exists e\ (\mathrm{inc}(e, u, v) \wedge \mathrm{lab}_b(e) \wedge \mathrm{Form}(\exists(a, c')))\\
&\mathrm{Form}(\exists([u = v]a, c')) && = (u = v \wedge \mathrm{Form}(\exists(a, c')))\\
&\mathrm{Form}(\exists([e = e']a, c')) && = (e = e' \wedge \mathrm{Form}(\exists(a, c')))
\end{aligned}$$

where the morphisms $[vb]a$, $[euvb]a$, $[u = v]a$, $[e = e']a$ can be decomposed into some morphism $a\colon P \to C$ after $[vb]P \to P$ (addition of a node labeled b), $[euvb]P \to P$ (addition of an edge labeled b to some nodes u and v), $[u = v]P \to P$ (identification of nodes), $[e = e']P \to P$ (identification of edges), respectively. Form is extended for the operators $\neg, \wedge$ as usual. Note, Form is not unique, but well defined.

Remark. For every graph condition c over P, $\mathrm{Free}(\mathrm{Form}(c)) \subseteq \mathrm{D}_P \subseteq \mathrm{Var}$, i.e. all free variables of the constructed formula $\mathrm{Form}(c)$ correspond to elements in P.

Example 23. Let $\mathrm{C_V} = \{0, 1\}$ and $\mathrm{C_E} = \{b, d\}$. The graph condition

$$\forall(\emptyset \to \textcircled{0}_u,\ \exists(\textcircled{0}_u \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v))$$

with the meaning "For every node with label 0, there exists an outgoing b-labeled edge to a node with label 1" is transformed into the following first-order formula:

$$\begin{aligned}
&\mathrm{Form}(\neg\exists(\emptyset \to \textcircled{0}_u,\ \neg\exists(\textcircled{0}_u \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v)))\\
&= \neg\mathrm{Form}(\exists(\emptyset \to \textcircled{0}_u,\ \neg\exists(\textcircled{0}_u \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v)))\\
&= \neg\exists u\ (\mathrm{node}(u) \wedge \mathrm{lab}_0(u) \wedge \mathrm{Form}(\exists(\textcircled{0}_u \to \textcircled{0}_u,\ \neg\exists(\textcircled{0}_u \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v))))\\
&= \neg\exists u\ (\mathrm{node}(u) \wedge \mathrm{lab}_0(u) \wedge \mathrm{Form}(\neg\exists(\textcircled{0}_u \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v)))\\
&= \neg\exists u\ (\mathrm{node}(u) \wedge \mathrm{lab}_0(u) \wedge \neg\mathrm{Form}(\exists(\textcircled{0}_u \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v)))\\
&= \neg\exists u\ (\mathrm{node}(u) \wedge \mathrm{lab}_0(u) \wedge \neg\exists v\ (\mathrm{node}(v) \wedge \mathrm{lab}_1(v) \wedge \mathrm{Form}(\exists(\textcircled{0}_u\ \textcircled{1}_v \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v))))\\
&= \neg\exists u\ (\mathrm{node}(u) \wedge \mathrm{lab}_0(u) \wedge \neg\exists v\ (\mathrm{node}(v) \wedge \mathrm{lab}_1(v) \wedge \exists e\ (\mathrm{inc}(e, u, v) \wedge \mathrm{lab}_b(e)\\
&\qquad\qquad \wedge \mathrm{Form}(\exists(\textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v \to \textcircled{0}_u \xrightarrow[e]{b} \textcircled{1}_v)))))\\
&= \neg\exists u\ (\mathrm{node}(u) \wedge \mathrm{lab}_0(u) \wedge \neg\exists v\ (\mathrm{node}(v) \wedge \mathrm{lab}_1(v) \wedge \exists e\ (\mathrm{inc}(e, u, v) \wedge \mathrm{lab}_b(e) \wedge \mathrm{true})))\\
&\equiv \forall u\ ((\mathrm{node}(u) \wedge \mathrm{lab}_0(u)) \Rightarrow \exists v\ \exists e\ (\mathrm{node}(v) \wedge \mathrm{lab}_1(v) \wedge \mathrm{inc}(e, u, v) \wedge \mathrm{lab}_b(e))).
\end{aligned}$$

with the same meaning "For every node u with label 0, there is a node v with label 1 and an b-labeled edge from u to v."

The proof of Theorem 11 depends on the following lemma.

Lemma 5. For all conditions c over P and all graphs G we have: For all graph morphisms $m\colon P \to G$ and all assignments $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$ with $m = \sigma[\mathrm{D}_P]$,

$$m \models_{\mathcal{A}} c \text{ iff } G[\![\mathrm{Form}(c)]\!](\sigma) = \mathit{true}.$$

Proof. By structural induction.
Basis. For $c = \mathrm{true}$, we have $m \models \mathrm{true} \Leftrightarrow G[\![\mathrm{true}]\!](\sigma) = \mathit{true}$.
Hypothesis. Assume the statement holds for condition c'.
Step. For conditions of the form $c = \exists(a, c')$, the statement is proved by induction over the decomposition of morphisms. If the morphism a is the identity $\mathrm{id}\colon P \to P$, we observe for all morphisms $m\colon P \to G$, and all assignments $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$ with $m = \sigma[\mathrm{D}_P]$, $m \models_{\mathcal{A}} \exists(\mathrm{id}, c') \Leftrightarrow m \models_{\mathcal{A}} c' \Leftrightarrow G[\![\mathrm{Form}(c')]\!](\sigma) = \mathit{true}$ (hypothesis for c').

Assume, the statement holds for a condition $\exists(a, c')$. We prove that the statement holds for all conditions $\exists(a', c')$ with extended morphism $a' = a[vb]$, $a[euvb]$, $a[u{=}v]$, and $a[e{=}e']$:

(1) $m \models_{\mathcal{A}} \exists([vb]a, c')$
$\Leftrightarrow$ $m \models_{\mathcal{A}} \exists([vb]P \to P,\ \exists(a, c'))$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $\exists m'.\ m = m' \circ [vb]P \to P$ and $m' \models_{\mathcal{A}} \exists(a, c')$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $\exists o \in \mathrm{D}_G.\ \sigma' = \sigma\{v/o\}$ and $G[\![\mathrm{node}(v)]\!](\sigma') = \mathit{true}$
and $G[\![\mathrm{lab}_b(v)]\!](\sigma') = \mathit{true}$ and $G[\![\mathrm{Form}(\exists(a, c'))]\!](\sigma') = \mathit{true}$ (Hyp., $\sigma'[\mathrm{D}_P] = m'$)
$\Leftrightarrow$ $G[\![\exists v\ (\mathrm{node}(v) \wedge \mathrm{lab}_b(v) \wedge \mathrm{Form}(\exists(a, c')))]\!](\sigma) = \mathit{true}$ (Def. $[\![\,]\!]$)
$\Leftrightarrow$ $G[\![\mathrm{Form}(\exists([vb]a, c'))]\!](\sigma) = \mathit{true}$ (Def. Form)
$\Leftrightarrow$ $G \models \mathrm{Form}(\exists([vb]a, c'))$ (Def. $\models$)

(2) $m \models_{\mathcal{A}} \exists([euvb]a, c')$
$\Leftrightarrow$ $m \models_{\mathcal{A}} \exists([euvb]P \to P,\ \exists(a, c'))$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $\exists m'.\ m = m' \circ [euvb]P \to P$ and $m' \models_{\mathcal{A}} \exists(a, c')$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $\exists o \in \mathrm{D}_G.\ \sigma' = \sigma\{e/o\}$ and $G[\![\mathrm{inc}(e, u, v)]\!](\sigma') = \mathit{true}$
and $G[\![\mathrm{lab}_b(e)]\!](\sigma') = \mathit{true}$ and $G[\![\mathrm{Form}(\exists(a, c'))]\!](\sigma') = \mathit{true}$ (Hyp., $\sigma'[\mathrm{D}_P] = m'$)
$\Leftrightarrow$ $G[\![\exists e\ (\mathrm{inc}(e, u, v) \wedge \mathrm{lab}_b(e) \wedge \mathrm{Form}(\exists(a, c')))]\!](\sigma) = \mathit{true}$ (Def. $[\![\,]\!]$)
$\Leftrightarrow$ $G[\![\mathrm{Form}(\exists([euvb]a, c'))]\!](\sigma) = \mathit{true}$ (Def. Form)
$\Leftrightarrow$ $G \models \mathrm{Form}(\exists([euvb]a, c'))$ (Def. $\models$)

(3) $m \models_{\mathcal{A}} \exists([u{=}v]a, c')$
$\Leftrightarrow$ $m \models_{\mathcal{A}} \exists([u{=}v]P \to P,\ \exists(a, c'))$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $\exists m'.\ m = m' \circ [u{=}v]P \to P$ and $m' \models_{\mathcal{A}} \exists(a, c')$ (Def. $\models_{\mathcal{A}}$)
$\Leftrightarrow$ $\sigma(u) = \sigma(v)$ and $G[\![u = v]\!](\sigma) = \mathit{true}$
and $G[\![\mathrm{Form}(\exists(a, c'))]\!](\sigma) = \mathit{true}$ (Hyp., $\sigma[\mathrm{D}_P] = m'$)
$\Leftrightarrow$ $G[\![(u = v \wedge \mathrm{Form}(\exists(a, c')))]\!](\sigma) = \mathit{true}$ (Def. $[\![\,]\!]$)
$\Leftrightarrow$ $G[\![\mathrm{Form}(\exists([u{=}v]a, c'))]\!](\sigma) = \mathit{true}$ (Def. Form)
$\Leftrightarrow$ $G \models \mathrm{Form}(\exists([u{=}v]a, c'))$ (Def. $\models$)

(4) $m \models_{\mathcal{A}} \exists([e{=}e']a, c') \Leftrightarrow G \models \mathrm{Form}(\exists([e{=}e']a, c'))$ (similar to (3)).

Since all identity-preserving graph morphisms may be decomposed into graph morphisms that subsequently add nodes, add edges, identify nodes and identify edges, the statement

holds for all conditions of the form $\exists(a, c')$. For conditions built with the operators $\neg$, $\wedge$, the proof of the statement is straightforward. □

Proof of Theorem 11. For all conditions c over $\emptyset$ and all graphs G, we have the following: $G \models_{\mathcal{A}} c$ iff for all morphisms $i_G\colon \emptyset \to G$, $i_G \models_{\mathcal{A}} c$ (Definition $\models_{\mathcal{A}}$) iff for all assignments $\sigma\colon \mathrm{Var} \to \mathrm{D}_G$, $G[\![\mathrm{Form}(c)]\!](\sigma) = \mathit{true}$ (Lemma 5, for $P = \emptyset$ and $\mathrm{Free}(\mathrm{Form}(c)) = \emptyset = \mathrm{D}_P$) iff $G \models \mathrm{Form}(c)$ (Definition $\models$). □

By Theorems 10 and 11, we obtain the equivalence of graph conditions under $\mathcal{A}$-satisfiability and first-order graph formulas.

Corollary 7 ($\mathcal{A}$: equivalence of graph conditions and graph formulas). $\mathcal{A}$-satisfiable graph conditions and first-order graph formulas are expressively equivalent.

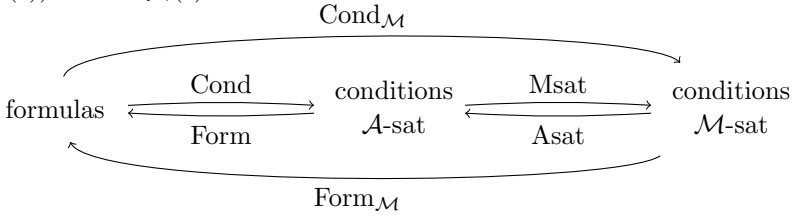
$$\text{formulas} \underset{\text{Form}}{\overset{\text{Cond}}{\rightleftarrows}} \text{conditions } \mathcal{A}\text{-sat}$$

The standard semantics for high-level conditions is $\mathcal{M}$-satisfiability. By Fact 4 and Theorems 1 and 2, $\mathcal{A}$-satisfiable graph conditions can be transformed into $\mathcal{M}$-satisfiable graph conditions and vice versa. Therefore, $\mathcal{M}$-satisfiable graph conditions and first-order graph formulas are expressively equivalent, as well.

Corollary 8 ($\mathcal{M}$: equivalence of graph conditions and graph formulas). $\mathcal{M}$-satisfiable graph conditions and first-order graph formulas are expressively equivalent.

$$\text{formulas} \underset{\text{Form}_{\mathcal{M}}}{\overset{\text{Cond}_{\mathcal{M}}}{\rightleftarrows}} \text{conditions } \mathcal{M}\text{-sat}$$

Proof. Immediate consequence of Theorems 10, 11 together with Theorems 1, 2, which allow to switch between $\mathcal{A}$- and $\mathcal{M}$-satisfiability: Define $\mathrm{Cond}_{\mathcal{M}} = \mathrm{Msat} \circ \mathrm{Cond}$ and $\mathrm{Form}_{\mathcal{M}} = \mathrm{Form} \circ \mathrm{Asat}$. By Theorems 10 and 1, for all formulas F over Σ and all graphs G, $G \models F \Leftrightarrow G \models_{\mathcal{A}} \mathrm{Cond}(F) \Leftrightarrow G \models \mathrm{Msat}(\mathrm{Cond}(F)) = \mathrm{Cond}_{\mathcal{M}}(F)$. By Theorems 11 and 2, for all formulas F over Σ and all graphs G, $G \models c \Leftrightarrow G \models_{\mathcal{A}} \mathrm{Asat}(c) \Leftrightarrow G \models \mathrm{Form}(\mathrm{Asat}(c)) = \mathrm{Form}_{\mathcal{M}}(c)$.



□

Conditions and formulas are *finite*, if the index set J of every conjunction $\wedge_{j\in J}$ and every disjunction $\vee_{j\in J}$ is finite. In the following, we want to strengthen the statement of Corollary 8 by showing that every finite graph condition can be translated into a finite

first-order graph formula and vice versa. We state additional requirements for a weak adhesive HLR category to ensure the effectiveness of the constructions in this paper.

Assumption 2. Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with
- a *finite number of epimorphisms for every given domain*, i.e., for all objects P, there exist only finitely many epimorphisms with domain P up to isomorphism, and
- a *finite number of matches*, i.e. for every morphism $l\colon K \hookrightarrow L$ and every object G, there exist only finitely many morphisms $m\colon L \to G$ up to isomorphism s.t. $\langle l, m \rangle$ has a pushout complement.

A finite number of matches implies finitely many morphisms between every pair of objects. The category (**Graphs**, *Inj*) satisfies the above requirements.We exemplarily show that Asat and Msat yield finite results for finite inputs.

Fact 13 (transformation of finite conditions). Let $\langle \mathcal{C}, \mathcal{M} \rangle$ be a weak adhesive HLR category with a finite number of epimorphisms for every given domain. For every finite condition c, the conditions $\mathrm{Msat}(c)$ and $\mathrm{Asat}(c)$ of Theorem 1 and 2 are finite, respectively.

Proof. For $\mathrm{Msat}(c)$ we have: As $\mathcal{C}$ has a finite number of epimorphisms for a given domain, the set $\mathcal{E}$ in the construction of $\mathrm{Msat}(e, c)$ is finite for every epimorphism e, and the set $\mathcal{E}$ in the construction of $\mathrm{Msat}(c)$ is finite. For $\mathrm{Asat}(c)$ we have: As $\mathcal{C}$ has a finite number of epimorphisms for a given domain, the set $\mathcal{E}$ in the construction of inM_C is finite for every object $C \in \mathcal{C}$, therefore $\mathrm{Asat}(c)$ is finite. □

Every finite graph condition can be translated into a finite first-order graph formula and vice versa.

Fact 14 (equivalence of finite graph conditions and finite graph formulas). $\mathcal{M}$-satisfiable finite graph conditions, $\mathcal{A}$-satisfiable finite graph conditions and finite first-order graph formulas are expressively equivalent.

Proof. By Corollaries 7 and 8, it suffices to show that all involved transformations preserve finiteness. For every finite graph condition c, the graph formula $\mathrm{Form}(c)$ is finite: For every graph morphism, there exists a finite decomposition, such that each morphism either adds a node, adds an edge, identifies two nodes, or identifies two edges. For every finite first-order graph formula F, the graph condition $\mathrm{Cond}(F)$ is finite: The disjunctions in the case $\mathrm{Cond}(X, \exists x\ F)$ are finite, as the label alphabet $\mathrm{C} = \mathrm{C_V} \cup \mathrm{C_E}$ is finite. As the category (**Graphs**, *Inj*) satisfies Assumption 2, we have by Fact 13: For every finite graph condition c, the graph conditions $\mathrm{Msat}(c)$ and $\mathrm{Asat}(c)$ are finite. □

By the undecidability of first-order graph formulas, we get that there are no effective procedures for deciding if a given graph condition is satisfiable at all, satisfied by every graph, nor if a given graph condition implies another given graph condition.

Corollary 9 (Undecidability of graph conditions). The satisfiability, the tautology, and the implication problem of (finite) graph conditions are undecidable.

Proof. Assume, the problems were decidable for (finite) graph conditions. By Corollary 8, we could construct for every (finite) first-order graph formula F_j $(i = 1, 2)$ a (finite) graph condition $\mathrm{Cond}_{\mathcal{M}}(F_j)$ such that $G \models F_j$ iff $G \models \mathrm{Cond}_{\mathcal{M}}(F_j)$. Then the satisfiability problem of (finite) first-order graph formulas would be decidable, contradiction (Trakhtenbrot 1950; Courcelle 1990). □

9. Conclusion

In this paper, the concepts of constraints and application conditions, investigated in e.g. (Heckel and Wagner 1995; Ehrig et al. 2006b; Koch et al. 2005), are unified and generalized to nested conditions over graph-like structures, along the lines of (Rensink 2004). As demonstrated by a case study on the handover protocol of mobile phones, graph conditions constitute a graphical and intuitive, yet precise formalism for specifying structural properties of system states. The results of this paper concern weak adhesive HLR sys-

Symbol	Description	Reference
Msat	From $\mathcal{A}$- to $\mathcal{M}$-satisfiability	Theorem 1
Asat	From $\mathcal{M}$- to $\mathcal{A}$-satisfiability	Theorem 2
B, N	(Interface) transformation of conditions	Lemma 2
Q	From $\mathcal{A}$- to $\mathcal{M}$-matching	Theorem 3
R	From $\mathcal{M}$- to $\mathcal{A}$-matching	Theorem 4
A	From constraints to application conditions	Theorem 5
L	From right to left application conditions	Theorem 6
$\mathrm{C}_\forall, \mathrm{C}_\exists$	From application conditions to constraints	Theorem 7
Appl	Construction of "gluing conditions"	Theorem 8
Cond	From formulas to graph conditions	Theorem 10
Form	From graph conditions to formulas	Theorem 11

Table 1. *Transformations of conditions and rules*

tems with nested application conditions and are listed in Table 1. Notable results are the transformations between $\mathcal{M}$- and $\mathcal{A}$-satisfiability of conditions, Msat and Asat, as well as the transformations between $\mathcal{A}$- and $\mathcal{M}$-matching of rules, R and Q, respectively. These transformations prove that if the monomorphism class $\mathcal{M}$ is strictly closed under decomposition, then the considered satisfiability and matching concepts are expressively equivalent. Otherwise, the notions of $\mathcal{M}$-satisfiability and $\mathcal{M}$-matching are more expressive. Moreover, the transformations can be applied to switch between the preferred notions of a user and the implemented notions of a rule-based transformation tool, and provide the basis for a data exchange between tools with different semantics. Finally, Msat and Asat enable the composition of complementary condition transformations in the case of $\mathcal{A}$-satisfiability, such as $\mathrm{L}_{\mathcal{A}}$.

A number of transformations between constraints and application conditions are investigated that can be used to develop correct transformation systems. This includes

transformations of constraints into application conditions, A, of right- into left application conditions and vice versa, L, and of application conditions in constraints, $C_\forall$ and $C_\exists$. Moreover the construction of an application condition (Appl) is considered that guarantees the applicability of the rule.

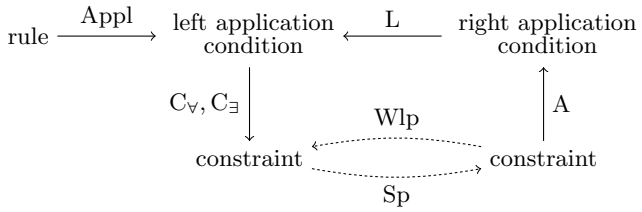


Figure 4. Basic transformations of conditions

The above basic transformations enable the construction of constraint-guaranteeing and constraint-preserving application conditions with the aim of restricting the applicability of rules to yield correct transformation systems. Moreover, the basic transformations are used for the construction of weakest preconditions (Dijkstra 1976; Dijkstra and Scholten 1989) and strongest postconditions with the aim to verify the correctness of transformation systems, see (Habel et al. 2006; Azab et al. 2006). For instance, if a given precondition implies the weakest precondition of a program relative to a postcondition, then the program in consideration is correct with respect to the pre- and postcondition (Habel et al. 2006).

Finally, transformations between graph conditions and first-order graph formulas are investigated. A main result of this paper is that both concepts are expressively equivalent, i.e. nested graph conditions capture exactly first-order graph properties. As a consequence, graph conditions inherit undecidability results for the satisfiability, tautology and implication problem. On the other hand, the transformation $\mathrm{Form}_{\mathcal{M}}$ of graph conditions to graph formulas enables the use of existing first-order automated theorem provers and satisfiability solvers.

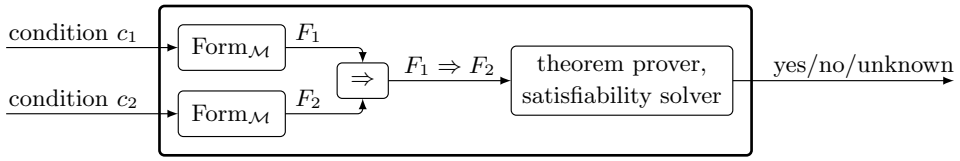


Figure 5. Semi-decider for the implication problem of $\mathcal{M}$-satisfiable graph conditions

Further topics may be the following.

(1) **Generalization of conditions.** Generalization of nested conditions to capture monadic second order properties. Consideration of temporal logic.

(2) **Extensions of the theory.** A consideration of conditions over partial and partially

labeled graphs. An investigation of weakest preconditions for rules with external interface (Pennemann 2008a).

(3) **Implementation.** The construction of weakest preconditions for transformation rules (via transformations $\mathrm{A}, \mathrm{L}, \mathrm{C}_\forall$ and Appl). The implementation of condition-based theorem provers (Orejas et al. 2008; Orejas 2008; Pennemann 2008b) and satisfiability solvers (Pennemann 2008a) to tackle the implication problem, and the application of existing theorem provers and satisfiability solvers (via $\mathrm{Form}_\mathcal{M}$) as a comparison. The implementation of semantic converters for rules and conditions to freely switch between $\mathcal{A}$-/$\mathcal{M}$-satisfiability and $\mathcal{A}$-/$\mathcal{M}$-matching), see (Azab et al. 2006; Zuckschwerdt 2006).

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