Attribution of Graphs by Composition of $\mathcal{M},\mathcal{N}$-adhesive Categories: Long Version

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Abstract. This paper continues the work on $\mathcal{M},\mathcal{N}$-adhesive categories and shows some important constructions on these categories. We use these constructions for an alternative, short proof for the $\mathcal{M},\mathcal{N}$-adhesiveness of partially labelled graphs. We further present a new concept of attributed graphs and show that the corresponding category is $\mathcal{M},\mathcal{N}$-adhesive. As a consequence, we inherit all nice properties for $\mathcal{M},\mathcal{N}$-adhesive systems such as the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem for this type of attributed graphs.

Keywords: Graph transformation, attributed graphs, composition, adhesive categories, adhesive systems

1 Introduction

The double-pushout approach to graph transformation, which was invented in the early 1970's, is the best studied framework for graph transformation [Roz97], [EEKR99, EKMR99, EEPT06b]. As applications of graph transformation come with a large variety of graphs and graph-like structures, the double-pushout approach has been generalized to the abstract settings of high-level replacement systems [EHKP91], adhesive categories [LS05], $\mathcal{M}$-adhesive categories [EGH10], $\mathcal{M},\mathcal{N}$-adhesive categories [HP12], and $\mathcal{W}$-adhesive categories [Gol12]. This paper continues the work of Habel and Plump [HP12] on $\mathcal{M},\mathcal{N}$-adhesive categories.

In the literature, there are several variants of attribution concepts, e.g. typed attributed graphs in the sense of Ehrig et al. [EEPT06b], attributed graphs in the sense of Plump [Plu09], attributed graphs as a graph with a marked sub-graph in the sense of Kastenberg and Rensink [KR12], separation of the graph structure and their attribution and data in the sense of Golas [Gol12], and attributed structures in the sense of Duval et al. [DEPR14].

Our main aim is to introduce a simple, alternative concept for attributed graphs and attributed graph transformation. Our approach is to define a category

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AttGraphs of attributed graphs from the well-known category **Graphs** of unlabelled graphs and a category **Att** of attribute collections by a multiset construction and the comma category construction. By closure results for $\mathcal{M}, \mathcal{N}$-adhesive categories, we obtain that the category **AttGraphs** is $\mathcal{M}, \mathcal{N}$-adhesive. By the results in [HP12], the Local Church-Rosser Theorem, the Parallelism Theorem and the Concurrency Theorem hold for the new type of attributed graphs provided that the HLR$^+$-properties are satisfied.

The paper is structured as follows. In Section 2, we recall the definition of $\mathcal{M}, \mathcal{N}$-adhesive categories. In Section 3, we prove some basic composition results and show that constructions for a string and a multiset category are $\mathcal{M}, \mathcal{N}$-adhesive for suitable classes $\mathcal{M}$ and $\mathcal{N}$ provided that the underlying category is. As a consequence, the category of partially labelled graphs is $\mathcal{M}, \mathcal{N}$-adhesive, as shown in Section 4. In Section 5, we introduce a new concept of attributed graphs - similar to partially labelled graphs - and show that the corresponding category of attributed graphs is $\mathcal{M}, \mathcal{N}$-adhesive. In Section 6, we present a precise relationship between $\mathcal{M}, \mathcal{N}$-adhesive and $\mathcal{W}$-adhesive categories, in Section 7 some related work, and in Section 8 some concluding remarks.

This paper is an extended version of the paper [PH15]. It contains additional examples and all proofs.

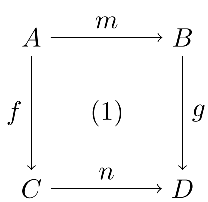
2 $\mathcal{M}, \mathcal{N}$-adhesive Categories

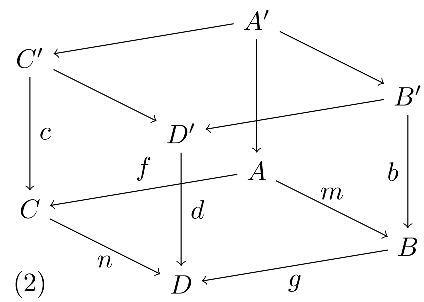
In this section, we recall the definition of $\mathcal{M}, \mathcal{N}$-adhesive categories, introduced in [HP12], generalizing the one of $\mathcal{M}$-adhesive categories [EGH10]. We assume that the reader is familiar with the basic concepts of category theory; standard references are [EEPT06b, Awo10].

Definition 1 ($\mathcal{M}, \mathcal{N}$-adhesive). A category **C** is $\mathcal{M}, \mathcal{N}$-adhesive, where $\mathcal{M}$ is a class of monomorphisms and $\mathcal{N}$ a class of morphisms, if the following properties are satisfied:

1. $\mathcal{M}$ and $\mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition. Moreover, $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, that is, $f; g \in \mathcal{N}$, $g \in \mathcal{M}$ implies $f \in \mathcal{N}$.
2. **C** has $\mathcal{M}, \mathcal{N}$-pushouts and $\mathcal{M}$-pullbacks. Also, $\mathcal{M}$ and $\mathcal{N}$ are stable under pushouts and pullbacks.
3. $\mathcal{M}, \mathcal{N}$-pushouts are $\mathcal{M}, \mathcal{N}$-van Kampen squares.

Remark. **C** has $\mathcal{M}, \mathcal{N}$*-pushouts*, if there is a pushout whenever one of the given morphisms is in $\mathcal{M}$ and the other morphism is in $\mathcal{N}$. **C** has $\mathcal{M}$*-pullbacks*, if there exists a pullback whenever at least one of the given morphisms is in $\mathcal{M}$. A class $\mathcal{X} \in \{\mathcal{M}, \mathcal{N}\}$ is *stable under* $\mathcal{M}, \mathcal{N}$*-pushouts* if, given the $\mathcal{M}, \mathcal{N}$-pushout (1) in the diagram below, $m \in \mathcal{X}$ implies $n \in \mathcal{X}$ and *stable under* $\mathcal{M}$*-pullbacks* if, given the $\mathcal{M}$-pullback (1) in the diagram below, $n \in \mathcal{X}$ implies $m \in \mathcal{X}$. An $\mathcal{M}, \mathcal{N}$-pushout is an $\mathcal{M}, \mathcal{N}$*-van Kampen square* if for the commutative cube (2) in the diagram below with the pushout (1) as bottom square, $b, c, d, m \in \mathcal{M}$, $f \in \mathcal{N}$, and the back faces being pullbacks, we have that the top square is a pushout if and only if the front faces are pullbacks.





In [HP12], it is shown that all $\mathcal{M}$-adhesive categories are $\mathcal{M},\mathcal{N}$-adhesive.

Lemma 1 ($\mathcal{M}$-adhesive $\Rightarrow$ $\mathcal{M},\mathcal{N}$-adhesive). Let $\mathbf{C}$ be any category and $\mathcal{N}$ be the class of all morphisms in $\mathbf{C}$. Then $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive if and only if $\mathbf{C}$ is $\mathcal{M}$-adhesive.

In the following, we give some examples of categories that are $\mathcal{M},\mathcal{N}$-adhesive.

Lemma 2 (Basic $\mathcal{M},\mathcal{N}$-adhesive Categories). The following categories are $\mathcal{M}$-adhesive [EEPT06b] and, by Lemma 1, $\mathcal{M},\mathcal{N}$-adhesive where $\mathcal{N}$ is the class of all morphisms in $\mathbf{C}$:

1. The category **Sets** of sets and functions is $\mathcal{M}$-adhesive where $\mathcal{M}$ is the class of all injective functions.
2. The category **Graphs** of graphs and graph morphisms is $\mathcal{M}$-adhesive where $\mathcal{M}$ is the class of all injective graph morphisms.
3. The category **LGraphs** of labelled graphs and graph morphisms is $\mathcal{M}$-adhesive where $\mathcal{M}$ is the class of all injective graph morphisms.

The following category is $\mathcal{M},\mathcal{N}$-adhesive, but not $\mathcal{M}$-adhesive [HP12]:

4. The category **PLGraphs** of partially labelled graphs and graph morphisms is $\mathcal{M},\mathcal{N}$-adhesive where $\mathcal{M}$ and $\mathcal{N}$ are the classes of all injective and all (injective) undefinedness-preserving[1] graph morphisms, respectively.

Finally, we define $\mathcal{M},\mathcal{N}$-adhesive systems as in [HP12].

Definition 2 ($\mathcal{M},\mathcal{N}$-adhesive Systems). Given an $\mathcal{M},\mathcal{N}$-adhesive category, a *rule* $\varrho = \langle p, \mathrm{ac}_L \rangle$ consists of a *plain rule* $p = \langle L \leftarrow K \rightarrow R \rangle$ with morphisms

[1] A morphism $f\colon G \to H$ *preserves undefinedness*, if it maps unlabelled items in G to unlabelled items in H.

$l\colon K \to L$ and $r\colon K \to R$ in $\mathcal{M}$, and an application condition ac_L over L in the sense of [HP09]. A *direct transformation* from an object G to an object H via the rule ϱ consists of two pushouts (1) and (2) as below where the vertical morphisms[2] are in $\mathcal{N}$ and $g \models \mathrm{ac}_L$. We write $G \Rightarrow_{\varrho,g} H$ if there exists such a direct transformation.

$$
\begin{array}{ccccc}
L & \xleftarrow{\;l\;} & K & \xrightarrow{\;r\;} & R \\
{\scriptstyle \mathrm{ac}_L \models\; g}\big\downarrow & (1) & \big\downarrow d & (2) & \big\downarrow h \\
G & \longleftarrow & D & \longrightarrow & H
\end{array}
$$

An $\mathcal{M},\mathcal{N}$-*adhesive system* consists of an $\mathcal{M},\mathcal{N}$-adhesive category and a set $\mathcal{R}$ of rules.

In [HP12], for $\mathcal{M},\mathcal{N}$-adhesive systems, three classical results of the double-pushout approach are proven: the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem.

3 Construction of Categories

There are various ways to construct new categories from given ones. Beside the standard constructions (product, slice and coslice, functor and comma category) we consider the constructions of a string category and a multiset category. For each of these constructions, we prove a composition result, saying more or less, whenever we start with $\mathcal{M}_i,\mathcal{N}_i$-adhesive categories, then the new constructed category is $\mathcal{M},\mathcal{N}$-adhesive for some $\mathcal{M},\mathcal{N}$. For the definitions of category-theoretic notions refer to [EEPT06b, Awo10].

First, we consider the standard constructions: product, slice and coslice, functor, and comma category. For the definitions we refer to [EEPT06b] A2 and A6. Our composition result generalizes the result from $\mathcal{M}$- to $\mathcal{M},\mathcal{N}$-adhesive categories.

Theorem 1 (Standard Constructions). $\mathcal{M},\mathcal{N}$-adhesive categories can be constructed as follows:

1. If $\mathbf{C_i}$ is $\mathcal{M}_i,\mathcal{N}_i$-adhesive ($i = 1, 2$), then the product category $\mathbf{C_1} \times \mathbf{C_2}$ is $\mathcal{M},\mathcal{N}$- adhesive where $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$.
2. If $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive and X is an object of $\mathbf{C}$, then the slice category $\mathbf{C}\backslash X$ and the coslice category $X\backslash\mathbf{C}$ over X are $\mathcal{M}',\mathcal{N}'$-adhesive where the morphism classes $\mathcal{M}',\mathcal{N}'$ are restricted to the slice and coslice category, i.e., for $\mathcal{X} \in \{\mathcal{M},\mathcal{N}\}$, $\mathcal{X}' = \mathcal{X} \cap \mathbf{C}\backslash X$ and $\mathcal{X}' = \mathcal{X} \cap X\backslash\mathbf{C}$, respectively.
3. If $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive, then for every category $\mathbf{X}$, the functor category $[\mathbf{X},\mathbf{C}]$ is $\mathcal{M}_{\mathrm{ft}},\mathcal{N}_{\mathrm{ft}}$-adhesive with functor transformations $\mathcal{M}_{\mathrm{ft}}$ and $\mathcal{N}_{\mathrm{ft}}$.[3]

[2] By stability of $\mathcal{N}$ under $\mathcal{M},\mathcal{N}$-pushouts, it is equivalent to require d in $\mathcal{N}$.

[3] For a class $\mathcal{X}$, $\mathcal{X}_{\mathrm{ft}}$ denotes the class of natural transformations $t\colon F \to G$, where all morphisms $t_X : F(X) \to G(X)$ are in $\mathcal{X}$.

4. If $\mathbf{C_i}$ are $\mathcal{M}_i, \mathcal{N}_i$-adhesive and $F_i\colon \mathbf{C_i} \to \mathbf{C}$ functors ($i = 1, 2$), where F_1 preserves $\mathcal{M}_1, \mathcal{N}_1$-pushouts and F_2 preserves $\mathcal{M}_2$-pullbacks, then the comma category $\mathbf{ComCat}(F_1, F_2, \mathcal{I})$ is $\mathcal{M}^c, \mathcal{N}^c$-adhesive where $\mathcal{M}^c = (\mathcal{M}_1 \times \mathcal{M}_2) \cap$ Mor, $\mathcal{N}^c = (\mathcal{N}_1 \times \mathcal{N}_2) \cap$ Mor, and Mor is the set of all morphisms of the comma category. We will use $A \downarrow B$ as a shorthand for the comma category $\mathbf{ComCat}(A, B, \mathcal{I})$, with $|\mathcal{I}| = 1$ and both functors A, B pointing into **Sets**.

Proof. The proof is a slight generalization of the corresponding one for $\mathcal{M}$-adhesive categories (see Theorem 4.15 in [EEPT06b]).

1. The product category $\mathbf{C}_1 \times \mathbf{C}_2$ is $\mathcal{M}, \mathcal{N}$-adhesive, because $\mathcal{M}$ and $\mathcal{N}$ inherit the required composition and decomposition properties from $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. Pushouts along $(\mathcal{M}, \mathcal{N})$-pairs of morphisms can be constructed componentwise. So can the van Kampen square, since pullbacks can also be constructed componentwise. The stability of $\mathcal{M}$ and $\mathcal{N}$ is inherited from pushouts and pullbacks in $\mathbf{C}_i$.
2. Morphisms, pullbacks and pushouts can similarly be constructed componentwise for both slice categories $\mathbf{C}\backslash X$ and coslice categories $X\backslash\mathbf{C}$. This construction also ensures that $\mathcal{M}'$ and $\mathcal{N}'$ are stable under pushout and pullback.
3. The functor category $[\mathbf{X}, \mathbf{C}]$, where $\mathcal{M}_{\text{ft}}$ and $\mathcal{N}_{\text{ft}}$ are functor transformations, is $\mathcal{M}, \mathcal{N}$-adhesive, because the functor transformations $\mathcal{M}_{\text{ft}}$ (or $\mathcal{N}_{\text{ft}}$) are monomorphisms in $[\mathbf{X}, \mathbf{C}]$ and the required composition and decomposition properties are inherited from $\mathcal{M}$ (or $\mathcal{N}$). Pushouts and pullbacks and the $\mathcal{M}, \mathcal{N}$-van Kampen square are constructed pointwise, i.e. for each object $X \in [\mathbf{X}, \mathbf{C}]$. The pointwise construction also ensures that $\mathcal{M}_{\text{ft}}$ and $\mathcal{N}_{\text{ft}}$ are stable under pushout and pullback.
4. The comma category $\mathbf{ComCat}(F_1, F_2, \mathcal{I})$ is $\mathcal{M}^c, \mathcal{N}^c$-adhesive, because $\mathcal{M}^c$ and $\mathcal{N}^c$ inherit the required composition and decompostion properties from $\mathcal{M}_i$ and $\mathcal{N}_i$. $\mathcal{M}^c, \mathcal{N}^c$-pushouts can be constructed componentwise, since F_1 preserves them. $\mathcal{M}^c$-pullbacks can be constructed componentwise, since F_2 preserves them. Consequently $\mathcal{M}^c, \mathcal{N}^c$-van Kampen squares can equally constructed componentwise. The componentwise construction ensures that $\mathcal{M}^c$ and $\mathcal{N}^c$ are stable under pushout and pullback.

$\square$

Example 1.

1. The category **Graphs** of graphs is isomorphic to the functor category $[\ \mathrm{E} \rightrightarrows \mathrm{V}\ , \mathbf{Sets}]$.
2. For the type graph TG, the category $\mathbf{Graphs}_{TG}$ of graphs typed over TG is isomorphic to the slice category $\mathbf{Graphs}\backslash TG$.
3. The category **HyperGraphs** of hypergraphs is isomorphic to the comma category $\mathbf{ComCat}(ID_{\mathbf{Sets}}, \Box^*, \mathcal{I})$ where $\Box^*\colon \mathbf{Sets} \to \mathbf{Sets}$ assigns to each set A and function f the free monoid A^* and the free monoid morphism f^*, respectively, and $\mathcal{I} = \{1, 2\}$.

4. The category **ElemNets** of of elementary Petri nets is isomorphic to the comma category $\mathbf{ComCat}(ID_{\mathbf{Sets}}, \mathcal{P}, \mathcal{I})$ where $\mathcal{P}\colon \mathbf{Sets} \to \mathbf{Sets}$ is the power set functor and $\mathcal{I} = \{1, 2\}$.
5. The category **PTNets** of of place/transition nets is isomorphic to the comma category $\mathbf{ComCat}(ID_{\mathbf{Sets}}, \Box^{\oplus}, \mathcal{I})$ where $\Box^{\oplus}\colon \mathbf{Sets} \to \mathbf{Sets}$ is the commutative monoid functor and $\mathcal{I} = \{1, 2\}$.

Second, we consider the constructions of a string and a multiset category and prove that $\mathcal{M}, \mathcal{N}$-adhesive categories are closed under these constructions.

Construction (String Category). Given a category $\mathbf{C}$, we construct a *string category* $\mathbf{C}^*$ as follows:

The objects are lists (finite sequences) $A_1 \dots A_m$ of objects of $\mathbf{C}$, including the empty list λ. The morphisms between two objects $A_1 \dots A_m$ and $B_1 \dots B_n$ (given $m \leq n$) are lists (finite sequences) of morphisms $f_1\colon A_1 \to B_i \dots f_m\colon A_m \to B_{i+m-1}$ in $\mathbf{C}$, with $B_1 \dots B_i \dots B_{i+m-1} \dots B_n$ for some $1 \leq i \leq m-n$ (i.e. $A_1 \dots A_m$ is *embedded* in $B_1 \dots B_n$). The empty list λ is an initial element for $\mathbf{C}^*$.

Our construction for a string category above is close to that of a free monoidal category. Allowing for the existence of a morphism even if $m < n$, however contradicts these definitions and further prevents us from giving a workable definition of a tensor product. We need these morphisms, especially in the case of the multiset category below, to allow for the addition or removal of elements in transformation systems based on these categories.

Construction (Multiset Category). Given a category $\mathbf{C}$, we construct a *multiset category* $\mathbf{C}^{\oplus}$ as follows:

The object are lists (finite sequences) $A_1 \dots A_m$ of objects of $\mathbf{C}$, including an empty list $\emptyset$. The morphisms between two objects $A_1 \dots A_m$ and $B_1 \dots B_n$ (given $m \leq n$) are lists (finite sequences) of morphisms $f_i\colon A_i \to B_{j_i}$ in $\mathbf{C}$, where $j_i = j_k$ implies $i = k$, $i \in \{1, \dots, m\}$.In contrast to the above construction for a string category, we ignore the order of elements.

We will use $\{\!|a, a, b|\!\}$ to denote a multiset with elements a, a and b.

Theorem 2 ($\mathbf{C}$ $\mathcal{M}$,$\mathcal{N}$-adh $\Rightarrow$ $\mathbf{C}^*$ $\mathcal{M}^*$,$\mathcal{N}^*$-adh, $\mathbf{C}^{\oplus}$ $\mathcal{M}^{\oplus}$,$\mathcal{N}^{\oplus}$-adh).

1. If $\mathbf{C}$ is $\mathcal{M}, \mathcal{N}$-adhesive, then the string category $\mathbf{C}^*$ over $\mathbf{C}$ is $\mathcal{M}^*, \mathcal{N}^*$-adhesive where $\mathcal{M}^*$ and $\mathcal{N}^*$ contain those morphisms which are lists of morphisms in $\mathcal{M}$ and $\mathcal{N}$, respectively. $\mathcal{N}^*$ is further restricted to morphisms that perserve length, i.e. where domain and codomain are of equal length.
2. If $\mathbf{C}$ is $\mathcal{M}, \mathcal{N}$-adhesive, then the multiset category $\mathbf{C}^{\oplus}$ over $\mathbf{C}$ is $\mathcal{M}^{\oplus}, \mathcal{N}^{\oplus}$-adhesive with $\mathcal{M}^{\oplus}$ and $\mathcal{N}^{\oplus}$ contain those morphisms which are lists of morphisms in $\mathcal{M}$ and $\mathcal{N}$, respectively.

Proof.

1. If $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive, then $\mathbf{C}^*$ is $\mathcal{M}^*,\mathcal{N}^*$-adhesive, because:

   $\mathcal{M}^*$ and $\mathcal{N}^*$ contain all isomorphisms, since an isomorphism in $\mathbf{C}^*$ must be a list of isomorphisms in $\mathbf{C}$ and $\mathcal{M}$ and $\mathcal{N}$, respectively contain all isomorphisms in $\mathbf{C}$. Isomorphisms must further be lists of equal length and $\mathcal{M}^*$ and $\mathcal{N}^*$ contain all morphisms with domain and codomain of equal length. Composition and decomposition properties can be inherited from $\mathbf{C}$, e.g. the composition of morphisms in $\mathcal{M}^*$ and $\mathcal{N}^*$ implies the composition of a list of morphisms in $\mathcal{M}$ and $\mathcal{N}$, respectively, which is possible since $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive. This also applies to the closure of $\mathcal{N}^*$ under $\mathcal{M}^*$-decomposition, since $f;g \in \mathcal{N}^*$ implies a domain and codomain of equal length, which in turn implies $f \in \mathcal{N}^*$.

   $\mathcal{M}^*,\mathcal{N}^*$-pushouts can be constructed componentwise if all morphisms preserve length, i.e. the pushout of lists in $\mathbf{C}^*$ is constructed from pushouts of its elements in $\mathbf{C}$.

$$\begin{array}{ccccc} A^m & \longrightarrow & B^m & \longrightarrow & B^n \\ \downarrow & (1) & \downarrow & (2) & \downarrow \\ C^m & \longrightarrow & D^m & \longrightarrow & D^n \end{array}$$

   If the horizontal morphisms do not preserve length, we can construct the pushout for the shorter lists first (diagram (1) above). B^m is B^n restricted to the domain of $f\colon A^m \to B^n$, resulting in an embedding $g\colon B^m \to B^n$. We construct D^n by replacing the occurence of B^m in B^n with D^n. Then (2) is a pushout, since for every object E^n with morphisms $x\colon B^n \to E^n$ and $y\colon D^m \to E^n$ there is a morphism $u_n\colon D^n \to E^n$ uniquely determined by the unique morphism $u_m\colon D^m \to E^m$ of pushout (1), with E^n constructed from E^m by adding the context originally omitted in B^m. By composition of pushouts $(1) + (2)$ is a pushout.

   $\mathbf{C}^*$ has all pullbacks, due to its inital element λ.

   $\mathcal{M}^*,\mathcal{N}^*$-pushouts are $\mathcal{M}^*,\mathcal{N}^*$-van Kampen squares, due to the componentwise construction.

2. If $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive, then $\mathbf{C}^\oplus$ is $\mathcal{M}^\oplus,\mathcal{N}^\oplus$-adhesive, because:

   $\mathcal{M}^\oplus$ and $\mathcal{N}^\oplus$ contain all isomorphisms, since an isomorphism in $\mathbf{C}^*$ must be a list of isomorphisms in $\mathbf{C}$ and $\mathcal{M}$ and $\mathcal{N}$, respectively contain all isomorphisms in $\mathbf{C}$. Composition and decomposition properties can be inherited from $\mathbf{C}$, e.g. the composition of morphisms in $\mathcal{M}^\oplus$ and $\mathcal{N}^\oplus$ implies the composition of a list of morphisms in $\mathcal{M}$ and $\mathcal{N}$, respectively, which is possible since $\mathbf{C}$ is $\mathcal{M},\mathcal{N}$-adhesive. $\mathcal{M}^\oplus,\mathcal{N}^\oplus$-pushouts can be constructed componentwise if all morphisms preserve length, i.e. the pushout of list in $\mathbf{C}^*$ is constructed from pushouts of its elements in $\mathbf{C}$.

$$
\begin{array}{ccccc}
A^m & \longrightarrow & B^m & \longrightarrow & B^n \\
\downarrow & (1) & \downarrow & (2) & \downarrow \\
C^m & \longrightarrow & D^m & \longrightarrow & D^n \\
\downarrow & (2') & \downarrow & (3) & \downarrow \\
C^o & \longrightarrow & D^o & \longrightarrow & D
\end{array}
$$

Analogously to the string category above, we can first construct pushout (1) componentwise, (2) and $2'$ are again constructed as in the string category above (although with the morphisms ignoring the order of elements). We can then construct D as D^m together with the elements of D^o and D^n that do not occur in D^m. Again, the morphism $u\colon D \to E$ is uniquely determined by the pushouts (2) and $(2')$. By composition of pushouts $(1)+(2)+(2')+(3)$ is a pushout.

$\mathbf{C}^\oplus$ has all pullbacks, due to its inital element λ.

$\mathcal{M}^\oplus, \mathcal{N}^\oplus$-pushouts are $\mathcal{M}^\oplus, \mathcal{N}^\oplus$-van Kampen squares, due to the componentwise construction.

□

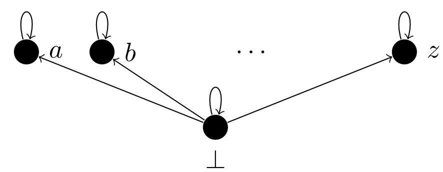
4 Partially Labelled Graphs

Let us reconsider the category **PLGraphs** of partially labelled graphs, investigated e.g. in [HP02, HP12], where the labelling functions for nodes and edges are allowed to be partial. In [HP12], it is shown that **PLGraphs** is not $\mathcal{M}$-adhesive, but $\mathcal{M}, \mathcal{N}$-adhesive if we choose $\mathcal{M}$ and $\mathcal{N}$ as the classes of all injective and all injective, undefinedness-preserving graph morphisms, respectively. In this section, we present an alternative proof of the statement: We show that the category **PLGraphs** can be constructed from the category **Graphs** and a category **PL** of labels by a multiset construction and the comma category construction.

First, we consider a label set L together with the symbol $\perp$ indicating undefinedness. As morphisms we use all identities as well as all morphisms from $\perp$ to a label in L.

Lemma 3 (PL is a Category). For each alphabet L, the class of all elements in $L \cup \{\perp\}$[4] as objects and all morphisms of the form $\perp \to x$ and $x \to x$ $(x \in L \cup \{\perp\})$ forms the category **PL** where the composition of $x \to y$ and $y \to z$ is $x \to z$ and the identity on x is $x \to x$.

[4] We assume that $\perp$ is not an element of L.



Proof. Follows directly from the definiton. □

It can be shown that the category **PL** is $\mathcal{M}, \mathcal{N}$-adhesive.

Lemma 4 (PL is $\mathcal{M}, \mathcal{N}$-adhesive). The category **PL** is $\mathcal{M}, \mathcal{N}$-adhesive where $\mathcal{M}$ and $\mathcal{N}$ are the classes of all morphisms and all identities, respectively.

Proof.

1. $\mathcal{M}$ and $\mathcal{N}$ contain all identity morphisms, which are the only isomorphisms in **PL**. They are also closed under composition and decompostion. Since $f; g \in \mathcal{N} \Rightarrow f, g \in \mathcal{N}$, $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition.
2. **PL** has $\mathcal{M}, \mathcal{N}$-pushouts: Since $\mathcal{N}$ contains only identity morphisms, there are only two cases, with either $\perp \to l$ or another identity morphism as the horizontal morphism:

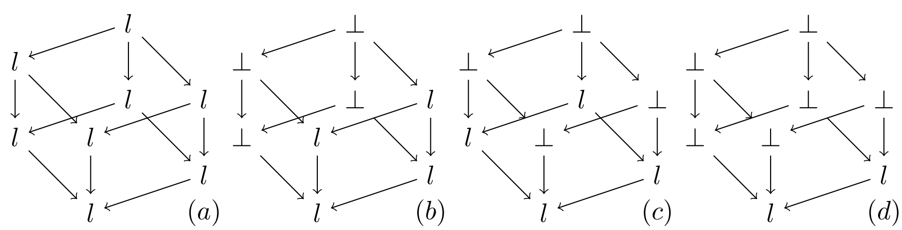
$$\begin{array}{ccc} l & \xrightarrow{m} & l \\ f\downarrow & (3) & \downarrow g \\ l & \xrightarrow[n]{} & l \end{array} \qquad\qquad \begin{array}{ccc} \perp & \xrightarrow{m} & l \\ f\downarrow & (4) & \downarrow g \\ \perp & \xrightarrow[n]{} & l \end{array}$$

   By of pushouts, the diagrams (3) and (4) are pushouts: the morphisms g, n are the only possible morphisms to obtain commutativity and the universal property holds. Since **PL** contains the initial element $\perp$, **PL** has all $\mathcal{M}$-pullbacks. $\mathcal{M}$ is trivially stable under pushouts and pullbacks, since $\mathcal{M}$ contains all morphisms. $\mathcal{N}$ is stable under pushouts and pullbacks because in the two cases above, the only possible morphisms f, g are identity morphisms and therefore in $\mathcal{N}$.
3. In **PL**, $\mathcal{M}, \mathcal{N}$-pushouts are $\mathcal{M}, \mathcal{N}$-van Kampen squares: Let (1) be a pushout, where $m \in \mathcal{M}$ and $f \in \mathcal{N}$. We have to show that, given a commutative cube (2) with (1) as bottom face, $b, c, d \in \mathcal{M}$ and pullbacks as back faces, the following holds:

   the top face is a pushout $\Leftrightarrow$ the front faces are pullbacks

   We have already identified the possible pushouts (3) and (4) above. These pushouts lead to four possible cases for back faces that are pullbacks:
   - For the pushout (3): The possible spans to construct the back faces of the cube with are:

    - $l \leftarrow l \rightarrow l$, which lead to the identity cube (a). Then all faces in the cube are pushouts as well as pullbacks and, therefore, constitute an $\mathcal{M},\mathcal{N}$-van Kampen square.
    - $\perp \leftarrow \perp \rightarrow \perp$, which leads to the cube (c) with pullbacks as back faces. Then the top face is the pushout (3) and the front faces are pullbacks.
    - $l \leftarrow \perp \rightarrow l$, $\perp \leftarrow \perp \rightarrow l$, for both of which at least one of the back faces will not be a pullback
- For the pushout (4): The possible spans to construct the back faces of the cube with are:
    - $\perp \leftarrow \perp \rightarrow l$, which leads to the cube (b) with pullbacks as back faces. Then the top face is the pushout (4) and the front faces are pullbacks.
    - $\perp \leftarrow \perp \rightarrow \perp$, which leads to the cube (d) with pullbacks as back faces. Then the top face is the pushout (3) and the front faces are pullbacks.
- In both cases it is possible to contruct different commutative cubes, but all of these do not have pushouts as top faces nor pullbacks as front faces.



□

Partially labelled graphs generalize labelled graphs [Ehr79].

Definition 3 (PLGraphs). A *partially labelled graph* is a system $G = (V, E, s, t, l)$ consisting of finite sets V and E of nodes and edges, source and target functions $s, t\colon E \rightarrow V$, and a partial labelling function $l\colon E + V \rightarrow L$ [5], where L is a fixed set of labels.

A *morphism* $g\colon G \rightarrow H$ between graphs G and H consists of two functions $g_V : V_G \rightarrow V_H$ and $g_E\colon E_G \rightarrow E_H$ that preserve sources, targets and labels, that is, $g_E; s_H = s_G; g_V$, $g_E; t_H = t_G; g_V$, and $l_H(g(x)) = l_G(x)$ for all x in $\mathrm{Dom}(l_G)$.

Fact 1. The class of partially labelled graphs and its morphisms constitute a category **PLGraphs**, where morphism composition is function composition and the identity is the identity function.

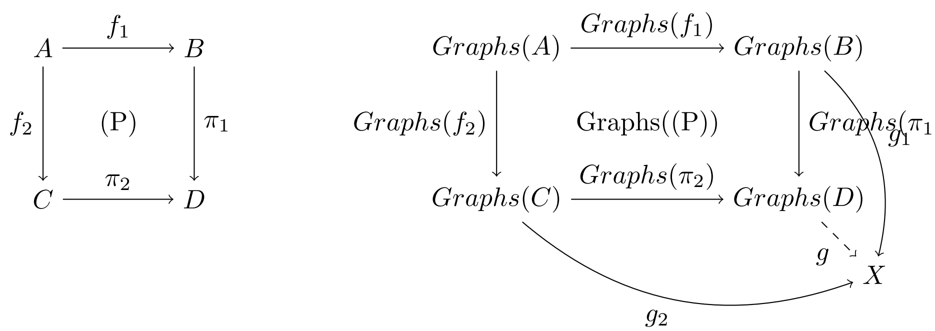
[5] + denotes the disjoint union of sets.

As an alternative to the existing proof we prove that the comma category of the two functors $Graphs\colon \mathbf{Graphs} \to \mathbf{Sets}$ and $PL\colon \mathbf{PL}^{\oplus} \to \mathbf{Sets}$ defined below is $\mathcal{M},\mathcal{N}$-adhesive. We further prove the category **PLGraphs** is isomorphic to this comma category, thus **PLGraphs** is also $\mathcal{M},\mathcal{N}$-adhesive. The isomorphism of categories is defined as in Ehrig et. al. [EEPT06b].

Definition 4 ($Graphs\colon$ Graphs $\to$ Sets). The functor $Graphs\colon \mathbf{Graphs} \to \mathbf{Sets}$ maps graphs to their underlying set of nodes and edges and is given as follows: For a graph $G' = (V', E', s', t')$, let $Graphs(G') = V' + E'$ and for a graph morphism $f_{G'}\colon A \to B$, let $Graphs(f_{G'})$ be a natural transformation, defined by $Graphs(f)(x) = f_{V'}(x)$ if $x \in V'$ and $f_{E'}(x)$ otherwise.

Lemma 5. The functor $Graphs\colon \mathbf{Graphs} \to \mathbf{Sets}$ preserves $\mathcal{M},\mathcal{N}$-pushouts, where $\mathcal{M}$ is the class of injective graph morphisms, $\mathcal{N}$ is the class of all morphisms.

Proof. Given a pushout (P) in **Graphs**, we have to show that $Graphs((P))$ is a pushout in **Sets**, i.e. for every pair of commuting morphisms $g_1\colon Graphs(B) \to X$ and $g_2\colon Graphs(C) \to X$ there is a unique morphism $g\colon Graphs(D) \to X$.



Let $x \in D$, then either $x \in B$ or $x \in C$. If $x \in B$, then $g_1(x) \in X$, if otherwise $x \in C$ then $g_2(x) \in X$.

We can thus define $g(x) = \begin{cases} g_1(x) & \text{if } x \in B \\ g_2(x) & \text{otherwise} \end{cases}$

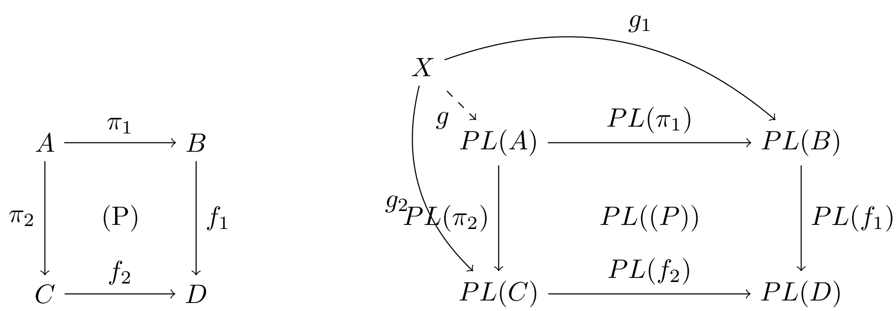
It remains to show that g is unique: The above construction follows directly from g_1, g_2 and $Graphs(\pi_1), Graphs(\pi_2)$. If we assume a different morphism $g'\colon Graphs(D) \to X$ it would have to follow these same constraints, hence $g' = g$. □

Definition 5 ($PL\colon \mathbf{PL}^{\oplus} \to$ Sets). The functor $PL\colon \mathbf{PL}^{\oplus} \to \mathbf{Sets}$ maps a multiset of labels to a set with distinct elements and is given as follows: For a multiset of labels $m' : L' \to \mathbb{N}$ let $PL(m) = \bigcup_{l' \in L'} \bar{m}(l')$, where for

$l' \in L'$, $\bar{m}(l') = \{l'_1, ..., l'_k\}$ iff $m(l') = k$. For a morphism $f\colon m_1 \to m_2$ let $PL(f) = PL(m_1) \to PL(m_2)$ be a morphism in **Sets**, such that $PL(f)(l'_1) = l'_2$ iff $PL(l_1) = l'_1, PL(l_2) = l'_2$ and $f(l_1) = l_2$ with $l_1, l_2 \in m_1, m_2$ respectively.

Lemma 6. The functor $PL\colon \mathbf{PL}^{\oplus} \to \mathbf{Sets}$ preserves $\mathcal{I}$-pullbacks, where $\mathcal{I}$ is the class of all morphisms.

Proof. Given a pullback (P) in $\mathbf{PL}^{\oplus}$, we have to show that $PL((P))$ is a pullback in **Sets**, i.e. for every pair of commuting morphisms $g_1\colon X \to PL(B)$ and $g_2\colon X \to PL(C)$ there is a unique morphism $g\colon X \to PL(A)$.



Let $x \in A$, then both $x \in B$ and $x \in C$ due to the morphisms π_1, π_2. We can thus define $g(x') = x$ if $g_1(x') = PL(\pi_1(x))$ with $x' \in X$.

It remains to show that g is unique: The above construction follows directly from g_1, g_2 and $PL(\pi_1), PL(\pi_2)$. If we assume a different morphism $g'\colon X \to PL(A)$ it would have to follow these same constraints, hence $g' = g$. □

For an object (G', m', op) of the comma category *Graphs* $\downarrow PL$ the morphism $\mathrm{op}\colon \mathit{Graphs}(G') \to PL(m')$ determines which node or edge is associated with which label, i.e. $\mathrm{op}(e') = l'_i$ with $e' \in E', l' \in L'$ and $i \in \mathbb{N}$ means the edge e' is labelled with l'.

Lemma 7 (PLGraphs $\cong$ *Graphs* $\downarrow_s PL$). The category **PLGraphs** of partially labelled graphs is isomorphic to the comma category *Graphs* $\downarrow_s PL$, where $\downarrow_s$ indicates a restriction to surjective morphisms op in the comma category.

We restrict ourselves to those objects of the comma category where op is surjective, since there could otherwise be labels that are not associated with an object in the graph.

Proof. The functor $F\colon \mathbf{PLGraphs} \to \mathit{Graphs} \downarrow_s PL$ is given by the following:

1. A partially labelled graph $G = (V, E, s, t, l)$ in **PLGraphs** is mapped to the triple (G', m', op) in *Graphs* $\downarrow_s PL$, where G', m' and op are defined

as follows: $G' = (V, E, s, t)$ is the graph without labels, $m' : L' \to \mathbb{N}$, where $L' = \text{Ran}(l)$ and $m'(l') = k$ with $l' \in L'$ and $k = |\{x | \forall x \in G : l(x) = l'\}|$ is a multiset of the labels of nodes and edges in G and $\text{op} : \mathit{Graphs}(G') \to PL(m')$ is a morphism in **Sets**, such that $\forall x' \in G' : \text{op}(\mathit{Graphs}(x')) = PL(l')$ iff $l(x) = l$ and $F(x) = x'$, $F(l) = l'$, where $x' \in G'$ is either an edge or a node in G'.
2. A morphism $f : A \to B$ in **PLGraphs** is mapped to a pair of a graph morphism $f_{G'} : G'_A \to G'_B$ and a morphism $f_{m'} : m'_A \to m'_B$ in $\mathbf{PL}^{\oplus}$, such that $\text{op}; PL(f_{L'}) = \mathit{Graphs}(f_{G'}); \text{op}$. The pair $(f_{G'}, f_{m'})$ constitutes a morphism in $\mathit{Graphs} \downarrow_s PL$.

Vice versa, the functor $F^{-1} : \mathit{Graphs} \downarrow_s PL \to$ **PLGraphs** is given by the following:

1. The triple $(G', m', \text{op} : \mathit{Graphs}(G') \to PL(L'))$ is mapped to a partially labelled graph $G = (V, E, s, t, l)$ as follows: $F^{-1} : (G') = (V, E, s, t)$ and $\text{op} : \mathit{Graphs}(G') \to PL(m')$ is mapped to l by: $\forall x \in G : l(x) = l'$ if $\text{op}(\mathit{Graphs}(x')) = PL(l')$ and $F^{-1}(x') = x$ and $l' \neq \perp$, $l(x)$ undefined otherwise, where $x \in G$ is either an edge or a node in G.
2. A morphism $f : A \to B$ in $\mathit{Graphs} \downarrow_s PL$ consists of a pair of a graph morphism $f_{G'} : G'_A \to G'_B$ and a morphism $f_{L'} : L'_A \to L'_B$ in $\mathbf{PL}^{\oplus}$.
   The graph morphism $f_G = f_{G'}$ is already a morphism in **PLGraphs**. Labels are preserved, since $\text{op}; \mathit{Graphs}(f_{L'}) = \mathit{Graphs}(f_{G'}); \text{op}$ holds, i.e the morphism $f_{G'}$ respects op, which determines labels in $\mathit{Graphs} \downarrow_s PL$.

It is easy to see that $F; F^{-1}$ are the identity functors on **PLGraphs** and $F^{-1}; F$ are the identity functors on $\mathit{Graphs} \downarrow_s PL$. This implies **PLGraphs** $\cong$ $\mathit{Graphs} \downarrow_s PL$. □

Now we are able to present an alternative proof of the fact that the category **PLGraphs** of partially labelled graphs is $\mathcal{M}, \mathcal{N}$-adhesive. It is based on fact that the categories **Graphs** of graphs and **PL** of labels are $\mathcal{M}, \mathcal{N}$-adhesive and the constructions of a commutative monoidal category and the comma category preserve $\mathcal{M}, \mathcal{N}$-adhesiveness.

Theorem 3 (PLGraphs is $\mathcal{M}, \mathcal{N}$-adhesive). The category **PLGraphs** of partially labelled graphs is $\mathcal{M}, \mathcal{N}$-adhesive where $\mathcal{M}$ and $\mathcal{N}$ are the classes of all injective and all injective, undefinedness-preserving graph morphisms, respectively.

Proof. The new proof of Theorem 3 is illustrated in Figure 1.

1. By Lemmata 2 and 4, **Graphs** and **PL** are $\mathcal{M}_G, \mathcal{N}_G$ and $\mathcal{M}_L, \mathcal{N}_L$-adhesive, respectively. $\mathcal{M}_G, \mathcal{N}_G$ are monomorphisms in **Graphs** and $\mathcal{M}_L, \mathcal{N}_L$ are the classes of all morphisms and all identity morphisms in **PL**, respectively.
2. By Theorem 2, $\mathbf{PL}^{\oplus}$ is $\mathcal{M}^{\oplus}, \mathcal{N}^{\oplus}$-adhesive.

3. By Theorem 1 and Lemmata 5 and 6, $Graphs \downarrow_s PL$ is $\mathcal{M}^c$,$\mathcal{N}^c$-adhesive. Note that Theorem 1 still holds for the restriction to a surjective op, since the componentwise constructions can be still be done just as in the unrestricted case.
4. By Lemma 7, **PLGraphs** is $\mathcal{M}$,$\mathcal{N}$-adhesive. Moreover $\mathcal{M} = F(\mathcal{M}^c)$ since both of these classes include all monomorphisms and $\mathcal{N} = F(\mathcal{N}^c)$ since the perservation of undefinedness in $\mathcal{N}$ is analogous to the restriction to identity morphisms in $\mathcal{N}^L$, which determines the treatment of labels in $\mathcal{N}^c$.

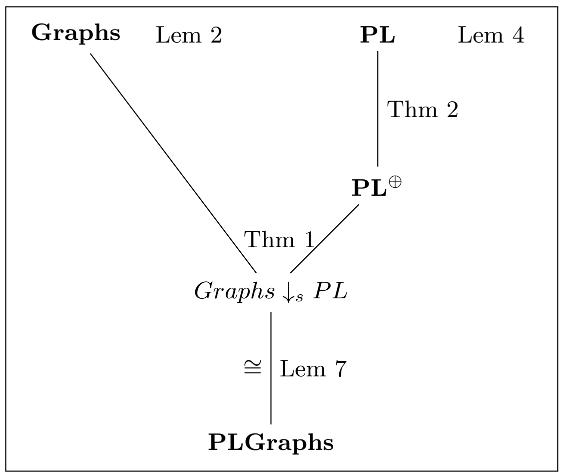


Fig. 1. Proof of "**PLGraphs** is $\mathcal{M}$,$\mathcal{N}$-adhesive".

□

Example 2. Figure 2 shows a transformation rule for a partially labelled graph consisting of a single node which is relabelled from a to b. Below the rule we show objects of $Graphs \downarrow PL$ and their individual components. Note that, in contrast to partially labelled graphs, we do not change the assigment of items to labels but instead change the label itself.

5 Attributed Graphs

Similiar to the construction for partially labelled graphs we construct attributed graphs, where the attributes can be changed analogously to relabelling.

We start with defining a category where the objects collect all the attributes of a node or an edge. These *attribute collections* consist of a set of names, each of which is associated with a value. We use the category **PL** from section 4 to represent these values.

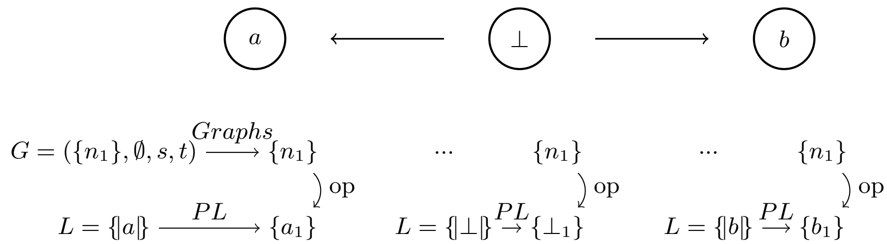


Fig. 2. Example transformation of an object of $Graphs \downarrow_s PL$

Definition 6 (Att $= ID_{\mathbf{Sets}} \downarrow PL$). The category **Att** of attribute collections is defined as the comma category $ID_{\mathbf{Sets}} \downarrow PL$ where $ID_{\mathbf{Sets}}$ denotes the identity functor over **Sets**.

Lemma 8 (Att is $\mathcal{M}^c$,$\mathcal{N}^c$-adhesive). The category **Att** of attribute collections is $\mathcal{M}^c$,$\mathcal{N}^c$-adhesive where $\mathcal{M}^c$,$\mathcal{N}^c$ are the classes of morphisms induced by the comma category construction.

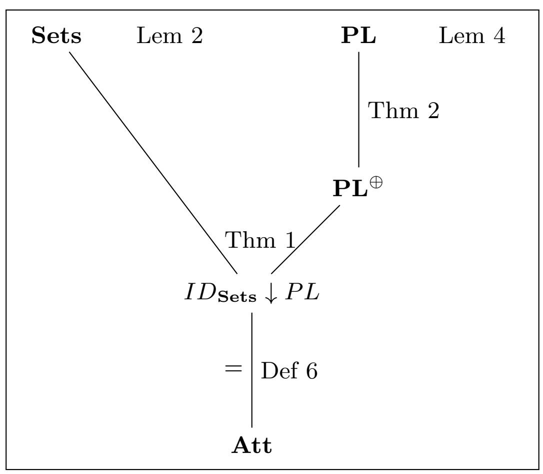


Fig. 3. Proof of "**Att** is $\mathcal{M}^c$,$\mathcal{N}^c$-adhesive".

Proof. The proof is illustrated in Figure 3. $ID_{\mathbf{Sets}} \downarrow PL$ is $\mathcal{M}^c$,$\mathcal{N}^c$-adhesive, since **Sets** and **PL** are $\mathcal{M}$,$\mathcal{N}$-adhesive and a multiset and comma category construction preserve $\mathcal{M}$,$\mathcal{N}$-adhesiveness. □

To construct attributed graphs we define a functor from multisets of these attribute collections to sets for later use in the comma category construction (compare with the construction for partially labelled graphs in section 4). We also prove that this functor preserves pullbacks, since this is required for the comma category to preserve $\mathcal{M},\mathcal{N}$-adhesiveness.

Definition 7 ($Att\colon \mathbf{Att}^{\oplus} \to \mathbf{Sets}$**).** The functor $Att\colon \mathbf{Att}^{\oplus} \to \mathbf{Sets}$ that maps attribute collections to sets with distinct values is given by the following: A triple $(ID_{\mathbf{Sets}}(S), PL(m), \mathrm{op})^{\oplus}$ is mapped to the set $S^{\oplus} + PL(m)^{\oplus}$ where $\oplus$ is flattened analogously to the way PL does (see Definition 5). A morphism $f\colon A \to B$ in $\mathbf{Att}^{\oplus}$ is mapped to a morphism $Att(f)\colon Att(A) \to Att(B)$, where elements in $Att(A)$ are mapped to elements in $Att(B)$ based on the original mappings in $\mathbf{Att}^{\oplus}$.

Lemma 9 (Att **preserves** $\mathcal{I}$**-pullbacks).** The functor $Att\colon \mathbf{Att}^{\oplus} \to \mathbf{Sets}$ preserves $\mathcal{I}$-pullbacks where $\mathcal{I}$ is the class of all morphisms.

Proof. In the same way that the defintion of Att is analogous to that of PL the preservation of pullbacks can be proven in a similar way. □

Example 3. Figure 4 shows a transformation rule in **Att**. The attribute collection consists of a single attribute a, which has its value changed from 5 to 9 by the rule. Below the rule we show the objects of **Att** with their individual components.

$$a = 5 \longleftarrow a = \bot \longrightarrow a = 9$$

$$\begin{array}{ccccccccc} N = \{a\} & \xrightarrow{ID_{\mathbf{Sets}}} & \{a\} & & \cdots & \{a\} & & \cdots & \{a\} \\ & & \downarrow \mathrm{op} & & & \downarrow \mathrm{op} & & & \downarrow \mathrm{op} \\ V = \{\!|5|\!\} & \xrightarrow{PL} & \{5_1\} & & V = \{\!|\bot|\!\} \xrightarrow{PL} & \{\bot_1\} & & V = \{\!|9|\!\} \xrightarrow{PL} & \{9_1\} \end{array}$$

Fig. 4. Example transformation rule for objects of **Att**

Definition 8 (AttGraphs $=$ $Graphs \downarrow Att$**).** The category **AttGraphs** of attributed graphs is defined as the comma category $Graphs \downarrow Att$.

Now we are able to show that the category **AttGraphs** of attributed graphs is $\mathcal{M},\mathcal{N}$-adhesive. It is based on the fact that the categories **Graphs** of graphs and **Att** of attribute collections are $\mathcal{M},\mathcal{N}$-adhesive and the constructions of a multiset category and a comma category preserve $\mathcal{M},\mathcal{N}$-adhesiveness.

Theorem 4 (AttGraphs is $\mathcal{M},\mathcal{N}$-adhesive). The category **AttGraphs** of attributed graphs is $\mathcal{M}^c,\mathcal{N}^c$-adhesive where $\mathcal{M}^c,\mathcal{N}^c$ are the classes of morphisms induced by the comma category construction.

Proof. The proof is illustrated in Figure 5.

1. By Lemmata 2 and 8, **Graphs** and **Att** are $\mathcal{M}_G,\mathcal{N}_G$ and $\mathcal{M}_A,\mathcal{N}_A$-adhesive, respectively. $\mathcal{M}_G,\mathcal{N}_G$ are monomorphisms in **Graphs** and $\mathcal{M}_A,\mathcal{N}_A$ are the classes of morphisms induced by the comma category construction of **Att**.
2. By Theorem 2, $\mathbf{Att}^{\oplus}$ is $\mathcal{M}^{\oplus},\mathcal{N}^{\oplus}$-adhesive.
3. By Theorem 1 and Lemmata 5 and 9, *Graphs* ↓ *Att* is $\mathcal{M}^c,\mathcal{N}^c$-adhesive.
4. By Defintion 8, **AttGraphs** is $\mathcal{M}^c,\mathcal{N}^c$-adhesive.

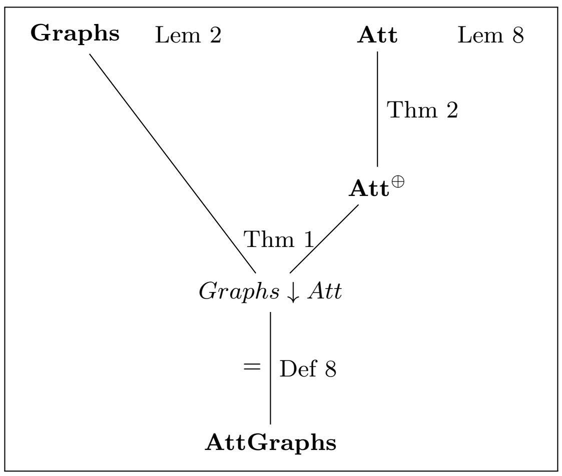


Fig. 5. Proof of "**AttGraphs** is $\mathcal{M}^c,\mathcal{N}^c$-adhesive".

□

In the following we briefly compare these attributed graphs to some existing attribution concepts. In contrast to the typed attributed graphs in [EEPT06b] these attributed graphs can have at most one value for an attribute. We constructed untyped graphs and even the attributes themselves have no types. Using the construction for the slice category from Theorem 1 we can build graphs where typing is done at either level, which allows us to have typed attributes on an untyped graphs, such that attribute values are constrained by the type but what attributes a node or edge has is not. In contrast typed attributed graphs require that the graph is typed and thus do not allow e.g. the addition of attributes to a node or edge. Compared to the attribution concepts used in [Plu09] we do not require a seperate instantiation of a rule schema and it is possible to find a match without fully specifying other, pontentially uninteresting, attributes. We do not, however base our attributes on an algebra that would allow us to perform some computations on the attributes, this would require additional work to prove $\mathcal{M}, \mathcal{N}$-adhesiveness for a suitable category. Fortunately we only need to provide this proof for such attributes once, enabling us to construct many different attributed structures without concerning ourselves with e.g. the underlying graphs.

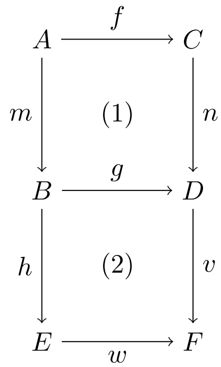
6 $\mathcal{W}$-adhesive Categories

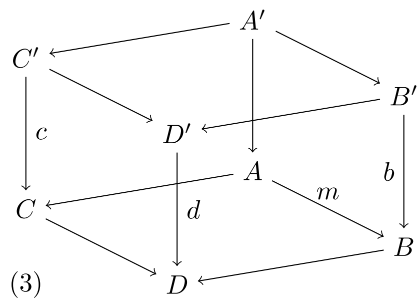
The concept of $\mathcal{M}, \mathcal{N}$-adhesive categories [HP12] was introduced as a framework for partially labeled graphs. More or less at the same time, the concept of $\mathcal{W}$-adhesive categories was introduced by Golas [Gol12] as a framework for attributed graphs. In this section, we present a precise relationship between $\mathcal{M}, \mathcal{N}$-adhesive and $\mathcal{W}$-adhesive categories.

First, we recall the definition of $\mathcal{W}$-adhesive categories.

Definition 9 ($\mathcal{W}$-adhesive). A tuple $(\mathbf{C}, \mathcal{R}, \mathcal{M}, \mathcal{W})$ is a *$\mathcal{W}$-adhesive category* if $\mathbf{C}$ is a category with morphism class $Mor_{\mathbf{C}}$, $\mathcal{R}$ and $\mathcal{M}$ are morphism classes with $\mathcal{R} \subseteq \mathcal{M}$, and $\mathcal{W} \subseteq \mathcal{R} \times Mor_{\mathbf{C}}$ is a class of morphism spans and

1. $\mathcal{M}$ is a class of monomorphisms closed under isomorphisms, composition, and decomposition, with $id_A \in \mathcal{M}$ for all $A \in \mathbf{C}$,
2. $\mathbf{C}$ has $\mathcal{M}$-pushouts and $\mathcal{M}$-pullbacks, as well as pushouts over $\mathcal{W}$-spans, called $\mathcal{W}$-pushouts,
3. $\mathcal{M}$ is stable under pullbacks, $\mathcal{R}$ is stable under $\mathcal{W}$-pushouts,
4. $\mathcal{W}$ is closed under $\mathcal{R}$, i.e., $(m' : A' \to B', a) \in \mathcal{W}$, $f' : A' \to C' \in \mathcal{R}$ implies $(f', a) \in \mathcal{W}$,
5. $\mathcal{W}$-pushout composition and decomposition: Given pushout (1), (1)+(2) is a $\mathcal{W}$-pushout with $(f, m; h) \in \mathcal{W}$ if and only if (2) is a $\mathcal{W}$-pushout with $(g, h) \in \mathcal{W}$,
6. $\mathcal{W}$-pushouts fulfill the $\mathcal{W}$-van Kampen property: Given a commutative cube (3) with $\mathcal{W}$-pushout (1) in the bottom, $m, d \in \mathcal{R}$, $b, c \in \mathcal{M}$ and the back faces being pullbacks, it holds that the top face is a pushout if and only if the front faces are pullbacks.





(3)

The classes $\mathcal{M}$ and $\mathcal{N}$ of horizontal and vertical morphisms naturally give rise to a class of $\mathcal{W}$-spans.

$\mathcal{W}$-adhesive categories might initially appear more expressive than $\mathcal{M}, \mathcal{N}$-adhesive categories, since the $\mathcal{W}$-spans can be limited to any subset of $\mathcal{R} \times Mor_{\mathbf{C}}$. The properties in Definition 9 however, limit the nature of $\mathcal{W}$-spans significantly.

Claim 1 ($\mathcal{W} = \mathcal{M} \times \mathcal{N}$). The class $\mathcal{W} \subseteq \mathcal{R} \times Mor_{\mathbf{C}}$ of all well-behaved spans is exactly the class $\mathcal{W} = \mathcal{R} \times \mathcal{N}$, where $\mathcal{R}$ denotes the class of the horizontal morphisms and $\mathcal{N}$ denotes the class of the vertical morphisms.

Proof of Claim 1 By Definition 9.4, $\mathcal{W}$-spans are closed under $\mathcal{R}$, i.e. for all $(a, b) \in \mathcal{W}$ and any $c \in \mathcal{R}$, $(c, b) \in \mathcal{W}$. Hence the class of all well-behaved spans $\mathcal{W}$ is exactly $\mathcal{R} \times \mathcal{N}$, where $\mathcal{N} = \{n | (a, n) \in \mathcal{W}\}$. □

Due to Claim 1, the class of $\mathcal{W}$-spans can also be seen as two classes $\mathcal{R}$ and $\mathcal{N}$. One could suspect that $\mathcal{R}$ and $\mathcal{N}$ form a $\mathcal{M}, \mathcal{N}$-adhesive category. As seen below, this is true up to stability of $\mathcal{N}$ over pushouts and pullbacks. The stability of $\mathcal{N}$ over pushouts and pullbacks is necessary for $\mathcal{N}$ to encompass all of the vertical morphisms of a derivation. Additionally it ensures the existence of pushouts in the proofs in [HP12].

We obtain the following relationship between $\mathcal{M}, \mathcal{N}$- and $\mathcal{W}$-adhesive categories.

Theorem 5 ($\mathcal{M}, \mathcal{N}$-adhesive $\Rightarrow$ $\mathcal{W}$-adhesive). If the category $\mathbf{C}$ is $\mathcal{M}, \mathcal{N}$-adhesive, then the tuple $(\mathbf{C}, \mathcal{M}, \mathcal{M}, \mathcal{M} \times \mathcal{N})$ is a $\mathcal{W}$-adhesive category. Vice versa, if the tuple $(\mathbf{C}, \mathcal{R}, \mathcal{M}, \mathcal{W})$ is $\mathcal{W}$-adhesive, then $\mathbf{C}$ is $\mathcal{R}, \text{Ran}(\mathcal{W})$-adhesive provided that the range $\text{Ran}(\mathcal{W})$ of $\mathcal{W}$ is stable under pushout and pullback and contains all isomorphisms.

Proof. By inspection of Definitions 1 and 9.

Let $\mathbf{C}$ be $\mathcal{M}, \mathcal{N}$-adhesive. Then we have:

1. By Definition 1.1, $\mathcal{M}, \mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition.

2. By definition 1.2, **C** has $\mathcal{M}$-pullbacks and $\mathcal{M},\mathcal{N}$-pushouts. With $\mathcal{W} = \mathcal{M} \times \mathcal{N}$, **C** has $\mathcal{W}$-pushouts.
3. By definition 1.2, $\mathcal{M}$ is stable under pullbacks and pushouts and $\mathcal{W} = \mathcal{M} \times \mathcal{N}$.
4. By $\mathcal{M}$ stable under pushout and $\mathcal{R} = \mathcal{M}$, $\mathcal{R}$ is stable under pushout.
5. **"If":** If (1) and (1)+(2) are $\mathcal{M},\mathcal{N}$-pushouts we have $m, m;h \in \mathcal{M}$ and $f \in \mathcal{N}$. Then (2) is an $\mathcal{M},\mathcal{N}$-pushout, since with $\mathcal{N}$ stable under pushout $g, w \in \mathcal{N}$ and since $\mathcal{M}$ is closed under decomposition $h \in \mathcal{M}$. With $\mathcal{R} = \mathcal{M}$ and $\mathcal{W} = \mathcal{R} \times \mathcal{N}$ all of these pushouts are also $\mathcal{W}$-pushouts.
   "Only If": If (1) and (2) are $\mathcal{M},\mathcal{N}$-pushouts, we have $m, h \in \mathcal{M}$ and $f, g \in \mathcal{N}$. Since $\mathcal{M}$ is closed under composition $m;h \in \mathcal{M}$. Then (1)+(2) is a $\mathcal{M},\mathcal{N}$-pushout and with $\mathcal{R} = \mathcal{M}$ and $\mathcal{W} = \mathcal{R} \times \mathcal{N}$ also a $\mathcal{W}$-pushout.
6. By $\mathcal{W} = \mathcal{M} \times \mathcal{N}$ the pushouts in the $\mathcal{M},\mathcal{N}$-van Kampen square are also $\mathcal{W}$-pushouts.

Thus, the tuple $(\mathbf{C}, \mathcal{M}, \mathcal{M}, \mathcal{M} \times \mathcal{N})$ is a $\mathcal{W}$-adhesive.

Let $(\mathbf{C}, \mathcal{M}, \mathcal{M}, \mathcal{M} \times \mathcal{N})$ be $\mathcal{W}$-adhesive and $\mathrm{Ran}(\mathcal{W})$ be stable under pushout and pullback and contain all isomorphisms. Then we have:

1. By Definition 9.1, $\mathcal{M}'$ is a class of monomorphisms closed under isomorphisms, composition and decomposition.
2. $\mathcal{N}$ is closed under composition and decomposition by Definition 9.5, which allows the composition and decomposition of two $\mathcal{W}$-pushouts along their vertical morphisms. $\mathcal{N}$ contains all isomorphisms by assumption.
3. By Definition 9.2, $(\mathbf{C}, \mathcal{R}, \mathcal{M}, \mathcal{W})$ has $\mathcal{M}$-pullbacks and $\mathcal{W}$-pushouts and with Claim 1 $\mathcal{M},\mathcal{N}$-pushouts.
4. By Definition 9.3, $\mathcal{M}'$ is stable over pushouts and pullbacks.
5. Stability of $\mathcal{N}$ over pushouts and pullbacks by assumption
6. By lemma 1 and appropriately chosen $\mathcal{M},\mathcal{N}$, the $\mathcal{W}$-pushouts in the van Kampen square are also $\mathcal{M},\mathcal{N}$-pushouts.

Thus, the category **C** is $\mathcal{M},\mathcal{N}$-adhesive. □

Remark. The situation may be summarized as follows:

- The requirements for an $\mathcal{M},\mathcal{N}$-adhesive category are slightly more strict than those for $\mathcal{W}$-adhesive categories.
- For $\mathcal{M},\mathcal{N}$-adhesive systems, the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem are proven. For $\mathcal{W}$-adhesive systems, up to our knowledge, there has only been a proof of (part of) the Local Church-Rosser Theorem.
- The $\mathcal{W}$-adhesive categories of attributed objects in [Gol12] are $\mathcal{M},\mathcal{N}$-adhesive: $\mathcal{N}$ is the class of $\perp$-preserving morphisms, contains all isomorphisms and is stable under pushout and pullbacks.

7 Related Concepts

Throughout the literature, various versions of adhesive and quasiadhesive [LS05], weak adhesive HLR [EHPP06], partial map adhesive [Hei10], and $\mathcal{M}$-adhesive [EGH10] exist. In [EGH10], all these categories are shown to be also $\mathcal{M}$-adhesive ones. The categories of labelled graphs, typed graphs, and typed attributed graphs in [EEPT06b], are known to be $\mathcal{M}$-adhesive categories if one chooses $\mathcal{M}$ to be the class of injective graph morphisms [EGH10]. Each such category induces a class of $\mathcal{M}$-adhesive systems for which several classical results of the double-pushout approach hold.

Unfortunately, the framework of $\mathcal{M}$-adhesive systems does not cover graph transformation with relabelling. In [HP12], the authors generalize $\mathcal{M}$-adhesive categories to $\mathcal{M},\mathcal{N}$-adhesive categories, where $\mathcal{N}$ is a class of morphisms containing the vertical morphisms in double-pushouts, and show that the category of partially labelled graphs is $\mathcal{M},\mathcal{N}$-adhesive, where $\mathcal{M}$ and $\mathcal{N}$ are the classes of injective and injective, undefinedness-preserving graph morphisms, respectively. Independently, Golas [Gol12] provided a general framework for attributed objects, so-called $\mathcal{W}$-adhesive systems which allows undefined attributes in the interface of a rule to change attributes, which is similar to relabelling. By Lemma 1 and Theorem 5, the hierarchy of adhesive categories in [EGH10] can be extended in the following way:

$$\text{adhesive} \underset{\not\Leftarrow}{\Rightarrow} \text{adhesive HLR} \underset{\not\Leftarrow}{\Rightarrow} \mathcal{M}\text{-adhesive} \underset{\not\Leftarrow}{\Rightarrow} \mathcal{M},\mathcal{N}\text{-adhesive} \underset{\not\Leftarrow}{\Rightarrow} \mathcal{W}\text{-adhesive}$$

In the literature, there are several variants of attribution concepts, e.g., Löwe et al. [LKW93] view graphs as a special case of algebras. These algebras can then additionally specify types for attributes. Ehrig et al. [EEPT06a] — introduce typed attributed graphs, expanding the graph by including an algebra for attribute values. To facilitate attribution, typed attributed graphs extend graphs by attribution nodes and attribution edges. All possible data values of the algebra are assumed to be part of the graph. Nodes and edges are attributed by adding an attribution edge that leads to an attribution node. Kastenberg and Rensink [KR12] take a similar approach, but instead of only encoding the data values, operations and constants are also included in the graph. Plump et al [Plu09] use a different approach to attribution. Here labels are replaced by sequences of attributes. Rules are complemented by rule schemata in which terms over the attributes are specified. These variables are substituted with attribute values and evaluated during rule application. Instead of modifying the definition of graphs and graph transformations to include attributes, Golas [Gol12] defines an attribution concept over arbitrary categories. Duval et al. [DEPR14] allow attributed graphs and allow rules to change attributes.

In [Peu13], Peuser compares the approaches of Ehrig et al. [EEPT06a] and Plump [Plu09] and introduces a useful new concept of attribution which is the basis of this work.

The idea of composition of adhesive categories is not new: For $\mathcal{M}$-adhesive categories, the standard constructions of product, slice and coslice, functor, and comma categories are given in [EEPT06b].

8 Conclusion

In this paper, we have continued the work on $\mathcal{M},\mathcal{N}$-adhesive categories and have presented several examples (see Table 1).

category	structures	adhesiveness	reference
Sets	sets	$\mathcal{M}$-adh	[EEPT06b]
PL	sets of labels	$\mathcal{M},\mathcal{N}$-adh	Lemma 4
Att	attribute collections	$\mathcal{M},\mathcal{N}$-adh	Lemma 8
Graphs	unlabelled graphs	$\mathcal{M}$-adh	[EEPT06b]
LGraphs	labelled graphs	$\mathcal{M}$-adh	[Ehr79]
PLGraphs	partially labelled graphs	$\mathcal{M},\mathcal{N}$-adh	[HP12], Thm 3
AttGraphs	attributed graphs	$\mathcal{M},\mathcal{N}$-adh	Theorem 4

Table 1. Examples of $\mathcal{M},\mathcal{N}$-adhesive categories

The main contributions of the paper are the following:

(1) Closure results for $\mathcal{M},\mathcal{N}$-adhesive categories.
(2) A new, shorter proof of the result in [HP12] that the category of partially labeled graphs is $\mathcal{M},\mathcal{N}$-adhesive.
(3) A new concept of attributed graphs together with a proof that the category of these attributed graphs is $\mathcal{M},\mathcal{N}$-adhesive and an application to transformation systems saying that for these attributed graphs, the Local Church-Rosser Theorem, the Parallelism Theorem and the Concurrency Theorem hold provided that the HLR$^+$-properties are satisfied.

Further topics might be:

(1) Investigate the relationship to the approach of Parisi-Presicce et al. [PEM87] considering graphs with a structured alphabet.
(2) Proof of the HLR$^+$-properties for the category **AttGraphs** to obtain the Local Church-Rosser Theorem, the Parallelism Theorem and the Concurrency Theorem for this type of attributed graphs.
(3) Generalization of the approach to systems with so-called left-linear rules, i.e., rules where only the left morphism of the rule is required to be in $\mathcal{M}$ as, e.g., in [BGS11].

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