$\mathcal{M},\mathcal{N}$-Adhesive Transformation Systems (Long Version)

Annegret Habel[1] and Detlef Plump[2]

[1] Carl von Ossietzky Universität Oldenburg
annegret.habel@informatik.uni-oldenburg.de
[2] The University of York
detlef.plump@york.ac.uk

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Abstract. The categorical framework of $\mathcal{M}$-adhesive transformation systems does not cover graph transformation with relabelling. Rules that relabel nodes are natural for computing with graphs, however, and are commonly used in graph transformation languages. In this paper, we generalise $\mathcal{M}$-adhesive transformation systems to $\mathcal{M},\mathcal{N}$-adhesive transformation systems, where $\mathcal{N}$ is a class of morphisms containing the vertical morphisms in double-pushouts. We show that the category of partially labelled graphs is $\mathcal{M},\mathcal{N}$-adhesive, where $\mathcal{M}$ and $\mathcal{N}$ are the classes of injective and injective, undefinedness-preserving graph morphisms, respectively. We obtain the Local Church-Rosser Theorem, the Parallelism Theorem and the Concurrency Theorem for graph transformation with relabelling and application conditions as instances of results which we prove at the abstract level of $\mathcal{M},\mathcal{N}$-adhesive systems.

Keywords: Graph transformation, relabeling, $\mathcal{M}$-adhesive categories, nested application conditions, Church-Rosser, Parallelism, Concurrency.

1 Introduction

The double-pushout approach to graph transformation, which was invented in the early 1970's, is the best studied framework for graph transformation [21, 5, 10, 4]. As applications of graph transformation come with a large variety of graphs and graph-like structures, the double-pushout approach has been generalised to the abstract settings of high-level replacement systems [9], adhesive categories [18] and $\mathcal{M}$-adhesive categories [8, 6, 7].

The categories of labelled graphs, typed graphs, and typed attributed graphs, for example, are known to be $\mathcal{M}$-adhesive categories if one chooses $\mathcal{M}$ to be the class of injective graph morphisms [8]. Each such category induces a class of $\mathcal{M}$-adhesive transformation systems for which several classical results of the double-pushout approach hold. Specifically, the Local Church-Rosser Theorem, the Parallelism Theorem, the Concurrency Theorem, the Amalgamation Theorem, the Embedding Theorem and the Local Confluence Theorem have been established for rules with nested application conditions [6, 7].

However, $\mathcal{M}$-adhesive transformation systems do not cover graph transformation systems with rules that relabel nodes. Such rules are natural for computing with graphs and are used as a foundation for the graph transformation language GP [19, 20]. The double-pushout approach can be extended with relabelling by introducing rules with partially labelled interface graphs [15], providing a theoretical foundation for graph transformation languages that is much simpler than attributed graph transformation in the sense of [4]. In the latter approach, attributed graphs contain the algebra underlying the operations in the attributes as well as special edges which connect nodes and edges with their attributes. Hence they are (usually) complex infinite objects which are difficult to comprehend and which do not directly correspond to the graph data structures used to implement graph transformation languages.

In this paper, we study transformation systems over the category PLG of partially labelled graphs and the class $\mathcal{M}$ of injective graph morphims (which are used in rules). It turns out that PLG violates two of the properties required for $\mathcal{M}$-adhesive categories: pushouts along $\mathcal{M}$-morphisms do not always exist and, when they exist, need not be pullbacks. We therefore generalise $\mathcal{M}$-adhesive categories to $\mathcal{M}, \mathcal{N}$-adhesive categories, where $\mathcal{N}$ is a class of morphisms containing the vertical morphisms in double-pushouts. $\mathcal{M}$-adhesive categories are then the special case where $\mathcal{N}$ is the class of all morphisms.

For $\mathcal{M}, \mathcal{N}$-adhesive transformation systems with (nested) application conditions, we prove three classical results of the double-pushout approach: the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem. We then show that PLG is $\mathcal{M}, \mathcal{N}$-adhesive, where $\mathcal{N}$ is the class of injective morphisms that preserve unlabelled nodes and edges. As a result, we obtain both theorems for the setting of graph transformation with relabelling and application conditions.

The paper is structured as follows. In Section 2, we generalise $\mathcal{M}$-adhesive categories to $\mathcal{M}, \mathcal{N}$-adhesive categories, prove that they satisfy the so-called HLR properties, and identify two additional factorization properties. In Section 3, we present the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem for $\mathcal{M}, \mathcal{N}$-adhesive transformation systems with application conditions. In Section 4, we show that the category PLG is $\mathcal{M}, \mathcal{N}$-adhesive for suitable classes $\mathcal{M}$ and $\mathcal{N}$ of morphisms. As a consequence, we obtain the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem for graph transformation with relabelling. In Section 5, we conclude and give some topics for future work.

This paper is an extended version of the paper [16]. It contains a subsection on concurrency, illustrating examples of all concepts, and includes all proofs.

2 $\mathcal{M}, \mathcal{N}$-Adhesive Categories

In [8] an overview is given on some categorical frameworks for double-pushout transformations. It is shown that adhesive categories [18], weak adhesive HLR categories [4], and partial map adhesive categories [17] are special cases of so-called $\mathcal{M}$-adhesive categories. A large number of results have been proved for $\mathcal{M}$-adhesive transformation systems, such as the Local Church-Rosser Theorem, the Parallelism Theorem, the Concurrency Theorem, the Amalgamation Theorem, the Embedding Theorem, and the Local Confluence Theorem [6, 7].

In this section, we generalize $\mathcal{M}$-adhesive categories as defined in [8, 6] to $\mathcal{M}, \mathcal{N}$-adhesive categories.

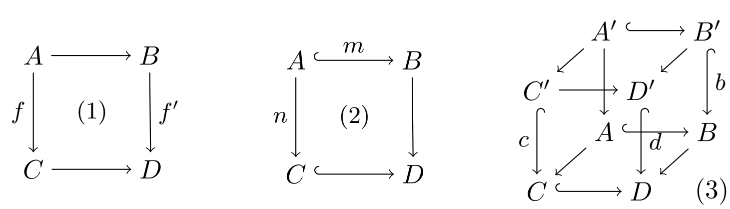
Definition 1 ($\mathcal{M}, \mathcal{N}$-adhesive category). A category $\mathcal{C}$ is $\mathcal{M}, \mathcal{N}$*-adhesive*, where $\mathcal{M}$ is a class of monomorphisms and $\mathcal{N}$ a class of morphisms, if the following properties are satisfied:

1. $\mathcal{M}$ and $\mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition (see [6]). Moreover, $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, that is, $g \circ f \in \mathcal{N}$, $g \in \mathcal{M}$ implies $f \in \mathcal{N}$.
2. $\mathcal{C}$ has $\mathcal{M}, \mathcal{N}$-pushouts and $\mathcal{M}$-pullbacks.
   Also, $\mathcal{M}$ and $\mathcal{N}$ are stable under pushouts and pullbacks.
3. $\mathcal{M}, \mathcal{N}$-pushouts are $\mathcal{M}, \mathcal{N}$-van Kampen squares (see below).

Remark 1. An $\mathcal{M}, \mathcal{N}$*-pushout* is a pushout where one of the given morphisms is in $\mathcal{M}$ and the other morphism is in $\mathcal{N}$. An $\mathcal{M}$*-pullback* is a pullback where at least one of the given morphisms is in $\mathcal{M}$.

A class $\mathcal{X}$ of morphisms is *stable under pushouts* if, given the pushout (1) in the diagram below, $f \in \mathcal{X}$ implies $f' \in \mathcal{X}$. Class $\mathcal{X}$ is *stable under pullbacks* if, given the pullback (1) in the diagram below, $f' \in \mathcal{X}$ implies $f \in \mathcal{X}$.

An $\mathcal{M}, \mathcal{N}$-pushout (2) with $m \in \mathcal{M}$ and $n \in \mathcal{N}$ is an $\mathcal{M}, \mathcal{N}$*-van Kampen square* if for any commutative cube (3) in the diagram below with the pushout (2) as bottom square, the vertical morphisms b, c, d in $\mathcal{M}$, and the back faces being pullbacks, we have that the top square is a pushout if and only if the front faces are pullbacks.



Fact 1. Let $\mathcal{C}$ be any category and let $\mathcal{N}$ be the class of all morphisms in $\mathcal{C}$. Then $\mathcal{C}$ is $\mathcal{M}, \mathcal{N}$-adhesive if and only if $\mathcal{C}$ is $\mathcal{M}$-adhesive in the sense of [6].

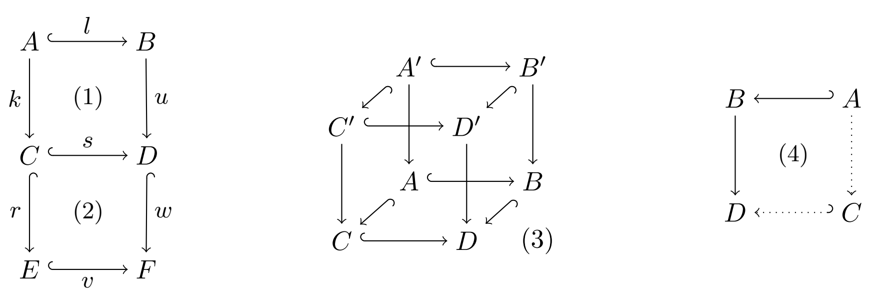
Proof. This follows from the definition of an $\mathcal{M}$-adhesive category because if $\mathcal{N}$ contains all morphisms, then $\mathcal{M},\mathcal{N}$-pushouts and $\mathcal{M},\mathcal{N}$-van Kampen squares are precisely the $\mathcal{M}$-pushouts and $\mathcal{M}$-van Kampen squares of [6], respectively. □

In Section 4, we show that the category PLG of partially labelled graphs is $\mathcal{M},\mathcal{N}$-adhesive but not $\mathcal{M}$-adhesive. In this case, $\mathcal{M}$ is the class of injective graph morphisms and $\mathcal{N}$ is the class of injective, undefinedness preserving graph morphisms.

$\mathcal{M},\mathcal{N}$-adhesive categories satisfy generalised versions of the so-called HLR-properties [9] of $\mathcal{M}$-adhesive categories.

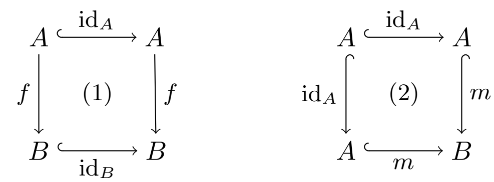
Theorem 1 (HLR-properties). Every $\mathcal{M},\mathcal{N}$-adhesive category satisfies the following *HLR-properties*:

1. $\mathcal{M},\mathcal{N}$-pushouts are pullbacks.
2. $\mathcal{M},\mathcal{N}$-pushout-pullback decomposition: If (1)+(2) in the diagram below is a pushout with $l \in \mathcal{M}$ and $r \circ k \in \mathcal{N}$ and (2) is a pullback with $w \in \mathcal{M}$, then (1) and (2) are pushouts as well as pullbacks.
3. $\mathcal{M},\mathcal{N}$-pullback decomposition: If (1)+(2) and (1) are pushouts with $l, w \in \mathcal{M}$ and $r \circ k \in \mathcal{N}$, then (1) and (2) are pullbacks.
4. Cube $\mathcal{M},\mathcal{N}$-pushout-pullback decomposition: If in the commutative cube (3) of the diagram below, all morphisms in the top square and in the bottom square are in $\mathcal{M}$, all vertical morphisms are in $\mathcal{N}$, the top square is a pullback, and the front faces are pushouts, then the bottom square is a pullback if and only if the back faces are pushouts.
5. Uniqueness of pushout complements: Given morphisms $A \hookrightarrow B$ in $\mathcal{M}$ and $B \rightarrow D$ in $\mathcal{N}$, there is, up to isomorphism, at most one object C with morphisms $A \rightarrow C$ and $C \hookrightarrow D$ such that (4) in the diagram below is a pushout.



We prove that $\mathcal{M},\mathcal{N}$-adhesive categories satisfy the HLR-properties, by generalising the corresponding proof for $\mathcal{M}$-adhesive categories in [4] (Theorem 4.26). We need the following fact.

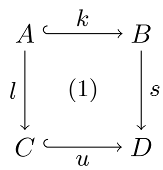
Fact 2. For every morphism $f\colon A \to B$, the square (1) below is a pushout and a pullback, and for every monomorphism $m\colon A \hookrightarrow B$, square (2) is a pullback.

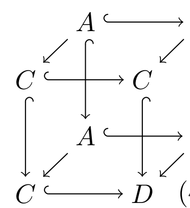


Proof. This follows from standard category theory. □

Proof of Theorem 1. We verify the HLR-properties.

1. *$\mathcal{M},\mathcal{N}$-pushouts are pullbacks.* Given an $\mathcal{M},\mathcal{N}$-pushout (1) with $k \in \mathcal{M}$ and $l \in \mathcal{N}$, we need to show that (1) is also a pullback.

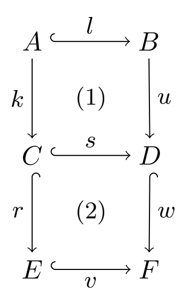


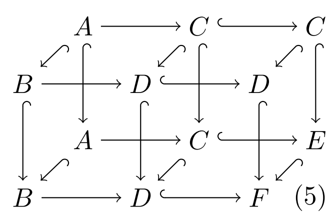


(4)

Consider the cube (4) with $\mathcal{M},\mathcal{N}$-pushout (1) as bottom square with the identities $\mathrm{id}_A, \mathrm{id}_C$. By Fact 2, the back faces are pullbacks and the top face is a pushout. Hence, by the $\mathcal{M},\mathcal{N}$-van Kampen property, the front faces, and therefore (1), are pullbacks.

2. *$\mathcal{M},\mathcal{N}$ pushout-pullback decomposition.* Consider the commutative diagram below (left), where (1)+(2) is a pushout with $l \in \mathcal{M}$ and $r \circ k \in \mathcal{N}$ and (2) is a pullback with $w \in \mathcal{M}$. We show that (1) and (2) are pushouts and also pullbacks.





(5)

Consider cube (5) with the identities $\mathrm{id}_B, \mathrm{id}_C, \mathrm{id}_D$. The bottom square, corresponding to square (1)+(2), is an $\mathcal{M},\mathcal{N}$-pushout and, by Definition 1, an $\mathcal{M},\mathcal{N}$-van Kampen square. By Fact 2 and $w \in \mathcal{M}$, the back faces and the left front face are pullbacks. By assumption, the right front face, corresponding to square (2), is a pullback. By the $\mathcal{M},\mathcal{N}$-van Kampen property, the top square, corresponding to square (1), is a pushout. By the pushout decomposition property [4], (1)+(2)

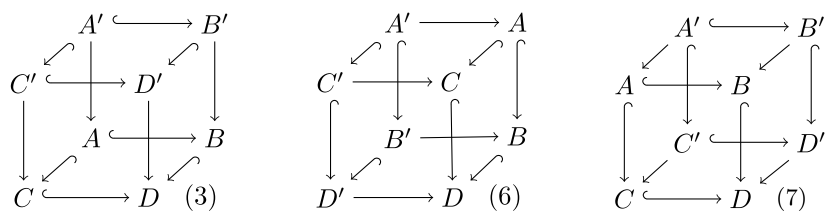
and (1) pushouts imply that (2) is a pushout. Since $\mathcal{M}$ is stable under pushouts, $l, w \in \mathcal{M}$ imply $s, v, r \in \mathcal{M}$. Since $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, $r \circ k \in \mathcal{N}$ and $r \in \mathcal{M}$ imply $k \in \mathcal{N}$. By Theorem 1.1, the $\mathcal{M}, \mathcal{N}$-pushout (1) is also a pullback.

3. *$\mathcal{M}, \mathcal{N}$ pullback decomposition.* Consider the commutative diagram above (left), where (1)+(2) and (1) are pushouts with $l, w \in \mathcal{M}$ and $r \circ k \in \mathcal{N}$. We show that (2) is a pullback. Since $\mathcal{C}$ has $\mathcal{M}$-pullbacks, we can construct a pullback object B' of $E \hookrightarrow F \hookleftarrow D$. The asising diagram is called (2'). By the universal pullback property, there is some morphism $B \to B'$ such that the arising diagram (1') and the triangle commute. Now, (1)+(2)=(1')+(2') is a pushout, (2') is a pullback. By $\mathcal{M}, \mathcal{N}$-pushout-pullback decomposition, the diagrams (1') and (2') are pushouts and pullbacks. By uniqueness of pushout complements, up to isomorphism, (1)=(1') and (2)=(2'), i.e. diagrams (1) and (2) are pullback.

4. *Cube $\mathcal{M}, \mathcal{N}$-pushout-pullback decomposition.* Consider the commutative cube (3) below, where all morphisms in the top and the bottom square are in $\mathcal{M}$, all vertical morphisms are in $\mathcal{N}$, the top square is a pullback and the front faces are $\mathcal{M}, \mathcal{N}$-pushouts. By Theorem 1.1, the front faces are also pullbacks. We show the following:

The bottom face is a pullback $\Leftrightarrow$ the back faces are pushouts.

"$\Rightarrow$" Let the bottom face in (3) be a pullback.

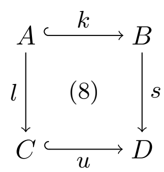


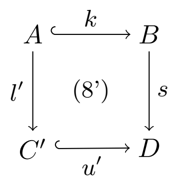
In cube (6), the following properties hold: The bottom is an $\mathcal{M}, \mathcal{N}$-pushout, by Theorem 1.1, a pullback and, by Definition 1, an $\mathcal{M}, \mathcal{N}$-van Kampen square. The front left face is an $\mathcal{M}, \mathcal{N}$-pushout and, by Theorem 1.1, a pullback. The back left face, corresponding to the top square in (3), is a pullback and the front right face, corresponding to the left front square in (3), is a pullback. By composition and decomposition of pullbacks [4], the back right face in (6) is a pullback. With the $\mathcal{M}, \mathcal{N}$-van Kampen property it follows that the top face in (6) is a pushout; this means that the back left face in (3) is a pushout. By turning the cube once more, we obtain the same result for the back right face in (3). Hence the back faces in (3) are pushouts.

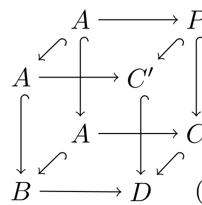
"$\Leftarrow$". Let the back faces in cube (3) be pushouts. By assumption, the top in cube (3) is a pullback, and the front faces are pushouts and, thus, pullbacks. By turning the cube again, we obtain cube (7), where the bottom, top, back left face, and the front right face are pushouts, and the back right face is a pullback.

Since the bottom square is an $\mathcal{M},\mathcal{N}$-van Kampen square and the top square is a pushout, the front faces are pullbacks; this means that the bottom square in cube (3) is a pullback.

5. *Uniqueness of pushout complements.* Let squares (8) and (8') below be pushouts, where $k\colon A \hookrightarrow B$ is in $\mathcal{M}$ and $s\colon B \to D$ is in $\mathcal{N}$. We show that C and C' are isomorphic. Consider cube (9), where P is the pullback object of $C' \hookleftarrow D \hookrightarrow C$. By the universal property of pullbacks, there is a unique morphism $A \to P$ such that the top and the front right face commute.







(9)

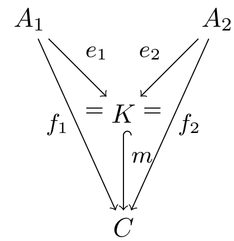
The bottom square in (9), corresponding to (8), is an $\mathcal{M},\mathcal{N}$-pushout and, therefore, an $\mathcal{M},\mathcal{N}$-van Kampen square. By Fact 2, the back left face is a pullback. By assumption, the front left face, corresponding to (8'), is a pushout and, therefore, a pullback. By construction, the front right face is a pullback, and, since pullbacks are closed under composition and decomposition, the back right face is a pullback. By the $\mathcal{M},\mathcal{N}$-van Kampen property, the top square is a pushout. Since pushouts preserve isomorphisms and id_A is an isomorphism, the morphism $P \hookrightarrow C'$ is an isomorphism. Analogously, by swapping the roles of C and C', we conclude that the morphism $P \hookrightarrow C$ is an isomorphism. Thus C and C' are isomorphic. $\square$

In order to prove the desired results for $\mathcal{M},\mathcal{N}$-adhesive transformation systems, three more properties will be needed.

Definition 2 (HLR$^+$**-properties).** Let $\mathcal{C}$ be an $\mathcal{M},\mathcal{N}$-adhesive category, $\mathcal{E}$ a class of morphisms, and $\mathcal{E}'$ a class of pairs of morphisms with the same codomain. Then the following properties are the HLR$^+$*-properties* with respect to $\mathcal{M}$, $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{E}'$.

1. $\mathcal{C}$ has binary coproducts, $\mathcal{M},\mathcal{E}$-pushouts and $\mathcal{M},\mathcal{E}$-pushouts are pullbacks.
2. $\mathcal{C}$ has an $\mathcal{E}'$-$\mathcal{N}$ *pair factorization* if, for each pair of morphisms (n_1, n_2) with morphisms $n_i\colon A_i \to C \in \mathcal{N}$ $(i = 1, 2)$, there exist (unique up to isomorphism) an object K and morphisms $e_i\colon A_i \to K$, $n\colon K \hookrightarrow C$ such that $n_i = n \circ e_i$, $(e_1, e_2) \in \mathcal{E}'$ and $n \in \mathcal{N}$ $(i = 1, 2)$.
3. $\mathcal{C}$ has an $\mathcal{E}'$-$\mathcal{M}$ *pair factorization* if, for each pair of morphisms (f_1, f_2) with morphisms $f_i\colon A_i \to C$ $(i = 1, 2)$, there exist (unique up to isomorphism) an object K and morphisms $e_i\colon A_i \to K$, $m\colon K \hookrightarrow C$ such that $f_i = m \circ e_i$

$(i = 1, 2)$, $(e_1, e_2) \in \mathcal{E}'$, and $m \in \mathcal{M}$.



Assumption 1. We assume that $\mathcal{C}$ is an $\mathcal{M},\mathcal{N}$-adhesive category and that $\mathcal{E}$ and $\mathcal{E}'$ are classes of morphisms and morphisms pairs, respectively, such that $\mathcal{C}$ satisfies the HLR$^+$-properties.

Binary coproducts are used in the definition of parallel rules (Definition 6). The $\mathcal{E}'$-$\mathcal{N}$ factorization and $\mathcal{C}$ has $\mathcal{M},\mathcal{E}$-pushouts and $\mathcal{M},\mathcal{E}$-pushouts are pullbacks are used in the proof of the Parallelism Theorem (Theorem 3), The $\mathcal{E}'$-$\mathcal{M}$ pair factorization is used in the proof of a shift lemma for application conditions (Lemma 1). The $\mathcal{E}'$-$\mathcal{M} \cap \mathcal{N}$ pair factorization is used in the construction of E-related transformations (Fact 4).

Example 1. The category PLG considered in Section 4 satisfies the HLR$^+$-properties, where $\mathcal{M}$ is the class of injective morphisms, $\mathcal{N}$ is the class of strongly undefinedness preserving morphisms, $\mathcal{E}$ is the class of surjective, strongly undefinedness preserving morphisms, and $\mathcal{E}'$ is the class of pairs of jointly surjective strongly undefinedness preserving morphisms.

3 $\mathcal{M},\mathcal{N}$-Adhesive Transformation Systems

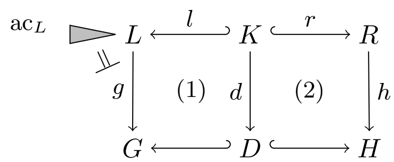
In this section, we introduce $\mathcal{M},\mathcal{N}$-adhesive transformation systems and present the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem in this setting.

We start by defining rules, direct transformations, and transformation systems.

Definition 3 (Rules, transformations, and systems). Given an $\mathcal{M},\mathcal{N}$-adhesive category, a *rule* $\varrho = \langle p, \mathrm{ac}_L \rangle$ consists of a *plain rule* $p = \langle L \hookleftarrow K \hookrightarrow R \rangle$ with morphisms $l\colon K \hookrightarrow L$ and $r\colon K \hookrightarrow R$ in $\mathcal{M}$, and an application condition ac_L over L (see below). A *direct transformation* from an object G to an object H via the rule ϱ consists of two pushouts (1) and (2) as below where the vertical morphisms[3] are in $\mathcal{N}$ and $g \models \mathrm{ac}_L$. We write $G \Rightarrow_{\varrho,g} H$ if there exists such a direct transformation. For a set of rules $\mathcal{R}$, we write $G \Rightarrow_{\mathcal{R}} H$, if $G \Rightarrow_{\varrho} H$ with

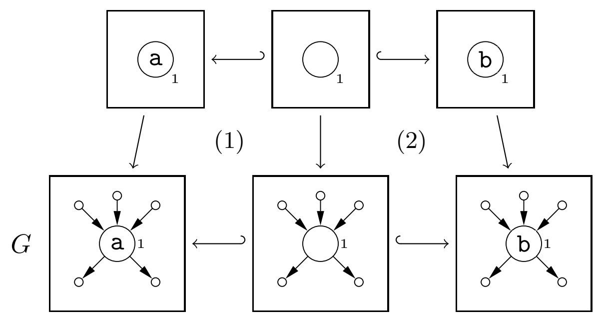
[3] By stability of $\mathcal{N}$ under $\mathcal{M},\mathcal{N}$-pushouts, it is equivalent to require d in $\mathcal{N}$.

ϱ in $\mathcal{R}$.



An $\mathcal{M},\mathcal{N}$*-adhesive transformation system* consists of an $\mathcal{M},\mathcal{N}$-adhesive category and a set $\mathcal{R}$ of rules.

Example 2. Consider the rule below for changing a label a into b with totally labelled the left- and right-hand side are partially labelled interface. Applying the rule to the node 1 in the graph G with a number of incoming and outgoing edges, the label of node 1 changes from a into b.

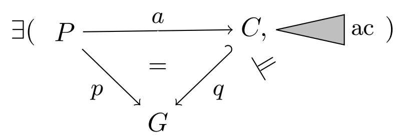


Remark 2. For rules $\varrho = \langle p, \mathrm{ac}_L, \mathrm{ac}_R \rangle$ with plain rule $p = \langle L \hookleftarrow K \hookrightarrow R \rangle$, left application condition ac_L, and right application condition ac_R [13], $\varrho^{-1} = \langle p^{-1}, \mathrm{ac}_R, \mathrm{ac}_L \rangle$ denotes the *inverse rule* of ϱ with plain rule $p^{-1} = \langle R \hookleftarrow K \hookrightarrow L \rangle$. For every direct transformation $G \Rightarrow_{\varrho,g,h} H$, there is a direct transformation $H \Rightarrow_{\varrho^{-1},h,g} G$.

Remark 3. Every $\mathcal{M}$-adhesive transformation system in the sense of [6] is an $\mathcal{M},\mathcal{N}$-adhesive transformation system if we choose $\mathcal{N}$ as the class of all morphisms in $\mathcal{C}$. Our notion of transformation system is more flemible because it allows to restrict the class of morphisms that are used to match rules. For example, one can show that every $\mathcal{M}$-adhesive category is $\mathcal{M},\mathcal{M}$-adhesive and hence gives rise to an $\mathcal{M},\mathcal{M}$-adhesive transformation system. A concrete example for this is the category of totally labelled graphs together with the class of injective graph morphisms (see also [12] for this setting).

Application conditions are nested constructs which can be represented as trees of morphisms equipped with quantifiers and Boolean connectives.

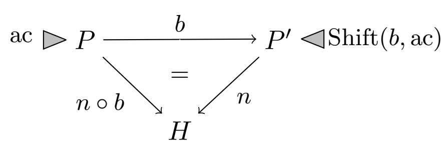
Definition 4 (Application condition). *Application conditions* are inductively defined as follows. For every object P, true is an application condition over P. For every morphism $a\colon P \to C$ and every application condition ac over C, $\exists(a, \mathrm{ac})$ is an application condition over P. For application conditions $\mathrm{ac}, \mathrm{ac}_i$ over P with $i \in I$ (for a given indem set I), $\neg\mathrm{ac}$ and $\wedge_{i\in I}\mathrm{ac}_i$ are application conditions over P. *($\mathcal{M}$-)satisfiability* of application conditions is also defined inductively. Every morphism satisfies true. A morphism $p\colon P \to G$ in $\mathcal{N}$ satisfies $\exists(a, \mathrm{ac})$ over P if there exists a morphism $q\colon C \to G$ in $\mathcal{M}$ such that $q \circ a = p$ and q satisfies ac.



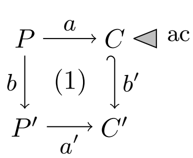
A morphism $p\colon P \to G$ in $\mathcal{N}$ satisfies $\neg\mathrm{ac}$ over P if p does not satisfy ac, and p satisfies $\wedge_{i\in I}\mathrm{ac}_i$ over P if p satisfies each ac_i ($i \in I$). We write $p \models \mathrm{ac}$ to empress that p satisfies ac.

Next we state two important technical results. The first lemma allows to shift application conditions over arbitrary morphisms. The proof makes use $\mathcal{E}'$-$\mathcal{M}$ pair factorization (which exists by the Assumption 1).

Lemma 1 (Shift of application conditions over morphisms [6]). There is a construction Shift such that, for each application condition ac over P and for each morphism $b\colon P \to P'$ in $\mathcal{N}$, Shift transforms ac via b into an application condition $\mathrm{Shift}(b, \mathrm{ac})$ over P' such that, for each morphism $n\colon P' \to H$ in $\mathcal{N}$, $n \circ b \models \mathrm{ac} \iff n \models \mathrm{Shift}(b, \mathrm{ac})$.



Construction. The Shift construction is inductively defined as follows:

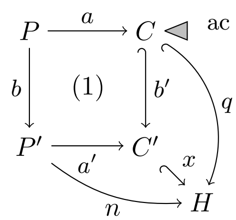


$\mathrm{Shift}(b, \mathrm{true}) = \mathrm{true}$.
$\mathrm{Shift}(b, \exists(a, \mathrm{ac})) = \bigvee_{(a',b')\in\mathcal{F}} \exists(a', \mathrm{Shift}(b', \mathrm{ac}))$ if $\mathcal{F} = \{(a', b') \in \mathcal{E}' \mid b' \in \mathcal{M}, (1) \text{ commutes}\} \neq \emptyset$ and false, otherwise. $\mathrm{Shift}(b, \neg\mathrm{ac}) = \neg\mathrm{Shift}(b, \mathrm{ac})$, $\mathrm{Shift}(b, \wedge_{i\in I}\mathrm{ac}_i) = \wedge_{i\in I}\mathrm{Shift}(b, \mathrm{ac}_i)$.

Proof. By structural induction. For the application condition true, the equivalence holds trivially. For an application condition of the form $\exists(a, \mathrm{ac})$ and morphisms $b\colon P \to P'$ in $\mathcal{N}$ and $n\colon P' \to H$ in $\mathcal{N}$, we have to show

$$n \circ b \models \exists(a, \mathrm{ac}) \iff n \models \mathrm{Shift}(b, \exists(a, \mathrm{ac})).$$

Only if. Let $n \circ b \models \exists(a, \mathrm{ac})$. By definition of satisfiability, there is some q in$\mathcal{M}$ with $q \circ a = n \circ b$ and $q \models \mathrm{ac}$. Since $\mathcal{N}$ is closed under composition, $n \circ b$ in $\mathcal{N}$. By $\mathcal{E}'$-$\mathcal{M}$-pair factorization, there exist an object C', morphisms $a' \colon P' \to C'$, $b' \colon C \to C'$ with (a', b') in $\mathcal{E}'$, and a morphism $x \colon C' \to H$ in $\mathcal{M}$ such that $y \circ a' = n$ and $y \circ b' = q$. Then $x \circ a' \circ b = n \circ b = q \circ a = x \circ b' \circ a$. Since $\mathcal{M}$ is closed under decomposition and x is in $\mathcal{M}$, we have $a' \circ b = b' \circ a$, i.e., (1) commutes. By q, x in $\mathcal{M}$, we have b' in $\mathcal{M}$. Thus, (a', b') is in $\mathcal{F}$. By induction hypothesis, $q = x \circ b' \models \mathrm{ac} \Leftrightarrow x \models \mathrm{Shift}(b', \mathrm{ac})$. Then $n = x \circ a' \models \exists(a', \mathrm{Shift}(b', \mathrm{ac}))$ and, by definition of Shift, $n \models \exists(b, \mathrm{Shift}(a, \mathrm{ac}))$.

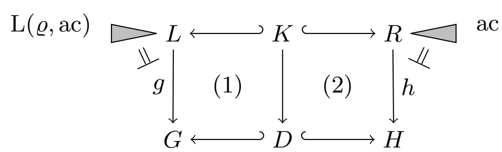


If. Let $n \models \mathrm{Shift}(b, \exists(a, \mathrm{ac}))$. Then there is some (a', b') in $\mathcal{F}$ such that b' in $\mathcal{M}$, $a' \circ b = b' \circ a$, and $n \models \exists(a', \mathrm{Shift}(b', \mathrm{ac}))$. By definition of satisfiability, there is some x in $\mathcal{M}$ such that $x \circ a' = n$ and $x \models \mathrm{Shift}(b', \mathrm{ac})$. By induction hypothesis, $x \models \mathrm{Shift}(b', \mathrm{ac}) \Leftrightarrow x \circ b' \models \mathrm{ac}$. Since $\mathcal{M}$ is closed under composition, $q = x \circ b'$ in $\mathcal{M}$. Thus, there is some $q \in \mathcal{M}$ such that $n \circ b = q \circ a \models \exists(a, \mathrm{ac})$.

For Boolean formulas over application conditions, the statement follows directly from the definition and the inductive hypothesis. Thus, the statement holds for all application conditions. □

The other technical result that we need is that application conditions can be shifted over rules.

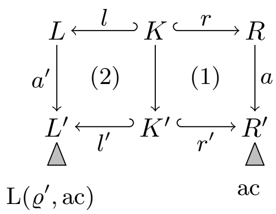
Lemma 2 (Shift of application conditions over rules [13]). There is a construction L such that, for each rule ϱ and each application condition ac over R, L transforms ac via ϱ into an application condition $\mathrm{L}(\varrho, \mathrm{ac})$ over L such that, for each direct transformation $G \Rightarrow_{\varrho,g,h} H$, we have $g \models \mathrm{L}(\varrho, \mathrm{ac}) \iff h \models \mathrm{ac}$.



Accordingly, there is a construction R such that, for each rule ϱ and each application condition ac over L, R transforms ac via ϱ into an application condition

$\mathrm{R}(\varrho, \mathrm{ac})$ over R such that, for each direct transformation $G \Rightarrow_{\varrho,g,h} H$, we have $h \models \mathrm{R}(\varrho, \mathrm{ac}) \iff g \models \mathrm{ac}$.

Construction. The L construction is inductively defined as follows:

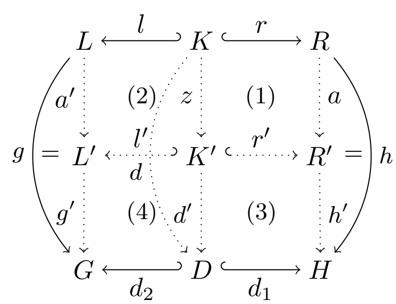


$\mathrm{L}(\varrho, \mathrm{true}) = \mathrm{true}$.
$\mathrm{L}(\varrho, \exists(a, \mathrm{ac})) = \exists(a', \mathrm{L}(\varrho', \mathrm{ac}))$ if $\langle r, a\rangle$ has a pushout complement (1) when $\varrho' = \langle L' \hookleftarrow K' \hookrightarrow R'\rangle$ is obtained by constructing the pushout (2), and false otherwise.
$\mathrm{L}(\varrho, \neg\mathrm{ac}) = \neg\mathrm{L}(\varrho, \mathrm{ac})$, $\mathrm{L}(\varrho, \wedge_{i\in I}\mathrm{ac}_i) = \wedge_{i\in I}\mathrm{L}(\varrho, \mathrm{ac}_i)$.
The R construction is defined by $\mathrm{R}(\varrho, \mathrm{ac}) = \mathrm{L}(\varrho^{-1}, \mathrm{ac})$.

Proof. By structural induction. For the right application condition true, the equivalence holds trivially. Assume now that the statement holds for right application conditions ac, that is,

$$(*) \quad \text{for all } G \Rightarrow_{\varrho',g',h'} H, \text{ we have } g' \models \mathrm{L}(\varrho', \mathrm{ac}) \Leftrightarrow h' \models \mathrm{ac}$$

Consider now a direct transformation $G \Rightarrow_{\varrho,g,h} H$ and an arbitrary right application condition of the form $\exists(a, \mathrm{ac})$.



Given a morphism $g'\colon L' \to G$ in $\mathcal{M}$ with $g' \circ a' = g$, there is a decomposition of the original pushouts into pushouts (1)–(4) as follows: Since $\mathcal{C}$ has $\mathcal{M}$-pullbacks and $D \to G$ is in $\mathcal{M}$, diagram (4) can be constructed as a pullback diagram of g' and $d_2\colon D \hookrightarrow G$ (in the figure above) and, since $\mathcal{M}$ is stable under pullbacks, $d'\colon K' \hookrightarrow D$ in $\mathcal{M}$. By the universal pullback property, there exists a unique morphism $z\colon K \to K'$ such that diagram (2) commutes and $d = d' \circ z$. By the $\mathcal{M},\mathcal{N}$-pushout-pullback decomposition, diagrams (2) and (4) are pushouts and pullbacks. Since $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, z in $\mathcal{N}$. Since $\mathcal{C}$ has $\mathcal{M},\mathcal{N}$-pushouts, diagram (1) can be constructed as a pushout over r in $\mathcal{M}$ and z is in $\mathcal{N}$. By the universal pushout property, there is a unique morphism $h'\colon R' \to H$ such that (3) commutes and $h' \circ a = h$. By pushout decomposition, (3) is a pushout and, since $\mathcal{M}$ is stable under pushouts, h' in $\mathcal{M}$. Vice versa, h' in $\mathcal{M}$ with $h' \circ a = h$ implies the existence of pushouts (1)–(4), starting by constructing diagram (3), such that g' is in $\mathcal{M}$ with $g' \circ b = g$. This may be summarized as follows:

$$(**)\quad \begin{array}{l} g' \text{ in } \mathcal{M}.\ g' \circ a' = g \text{ implies } \exists\, G \Rightarrow_{\varrho',g',h'} H.\ h' \text{ in } \mathcal{M}.\ h' \circ a = h \\ h' \text{ in } \mathcal{M}.\ h' \circ a = h \text{ implies } \exists\, G \Rightarrow_{\varrho',g',h'} H.\ g' \text{ in } \mathcal{M}.\ g' \circ a' = g \end{array}$$

By the Definitions of L and $\models$, statement (**), and the inductive hypothesis (*),

$$\begin{array}{l} \quad g \models \mathrm{L}(\varrho, \exists(a, \mathrm{ac})) = \exists(a', \mathrm{L}(\varrho', \mathrm{ac})) \\ \Leftrightarrow \exists\, g' \colon L' \to G \text{ in } \mathcal{M}.\ g' \circ a' = g \text{ and } g' \models \mathrm{L}(\varrho', \mathrm{ac}) \\ \Leftrightarrow \exists\, h' \colon R' \to H \text{ in } \mathcal{M}.\ h' \circ a = h \text{ and } h' \models \mathrm{ac} \\ \Leftrightarrow h \models \exists(a, \mathrm{ac}). \end{array}$$

Thus, for each direct derivation $G \Rightarrow_{\varrho,g,h} H$ and each right application condition of the form $\exists(a, \mathrm{ac})$, $g \models \mathrm{L}(\varrho, \exists(a, \mathrm{ac})) \Leftrightarrow h \models \exists(a, \mathrm{ac})$.

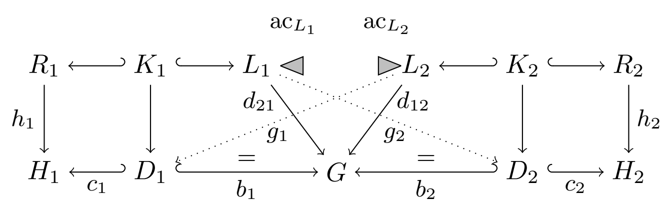
For Boolean formulas over right application conditions, the statement follows directly from the definition and the inductive hypothesis. Thus, the statement holds for all right application conditions.

By Definition of R and the property of L, for each direct transformation $G \Rightarrow_{\varrho,g,h} H$, $h \models \mathrm{R}(\varrho, \mathrm{ac}) \Leftrightarrow h \models \mathrm{L}(\varrho^{-1}, \mathrm{ac}) \Leftrightarrow g \models \mathrm{ac}$. □

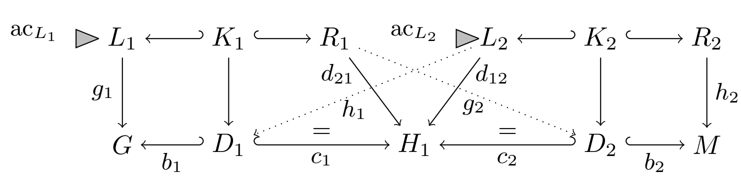
Assumption 2. For $i = 1, 2$, let $\varrho_i = \langle p_i, \mathrm{ac}_{L_i} \rangle$ be a rule with plain rule $p_i = \langle L_i \hookleftarrow K_i \hookrightarrow R_i \rangle$.

First, we formulate the notions of parallel and sequential independence and present the Local Church-Rosser Theorem.

Definition 5 (Parallel and sequential independence). Two direct transformations $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2} H_2$ are *parallelly independent* if in the diagram below there are morphisms $d_{ij} \colon L_i \to D_j$ such that $g_i = (b_j \circ d_{ij})$, $g'_i = (c_j \circ d_{ij}) \in \mathcal{N}$, and $g'_i \models \mathrm{ac}_{L_i}$ $(i, j \in \{1, 2\}$ and $i \neq j)$.

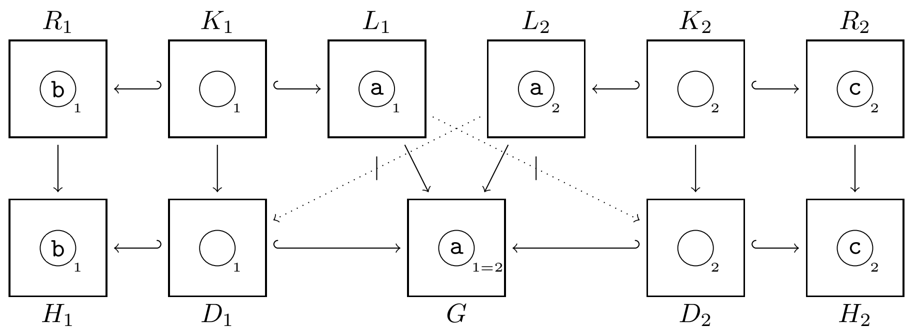


Two direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g_2} M$ are *sequentially independent* if in the diagram below there are morphisms $d_{12} \colon R_1 \to D_2$ and $d_{21} \colon L_2 \to D_1$ such that $h_1 = (c_2 \circ d_{12})$, $g_2 = (c_1 \circ d_{21})$, $h'_1 = (b_2 \circ d_{12}), g'_2 = (b_1 \circ d_{21}) \in \mathcal{N}$, $h'_1 \models \mathrm{R}(\varrho_1, \mathrm{ac}_{L_1})$, and $g'_2 \models \mathrm{ac}_{L_2}$.

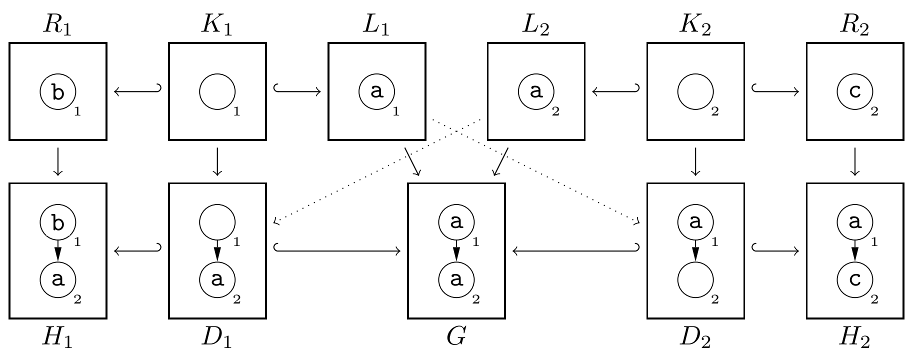


Remark 4. Parallel and sequential independence are closely related: Two direct transformations $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2} H_2$ are parallel independent if, and only if, the two direct transformations $H_1 \Rightarrow_{\varrho_1^{-1},h_1} G \Rightarrow_{\varrho_2,g_2} H_2$, where h_1 is the comatch of ϱ_1 in H_1, are sequentially independent.

Example 3. The direct transformations $H_1 \Leftarrow_{\varrho_1} G \Rightarrow_{\varrho_2} H_2$ below are not parallelly independent: there are no morphisms $d_{ij}\colon L_i \to D_j$ with the wanted properties. The corresponding direct transformations $H_1 \Rightarrow_{\varrho_1^{-1}} G \Rightarrow_{\varrho_2} H_2$ are not sequentially independent.



The direct transformations $H_1 \Leftarrow_{\varrho_1} G \Rightarrow_{\varrho_2} H_2$ below are parallelly independent: there are morphisms $d_{ij}\colon L_i \to D_j$ with the wanted properties. The corresponding direct transformations $H_1 \Rightarrow_{\varrho_1^{-1}} G \Rightarrow_{\varrho_2} H_2$ are sequentially independent.



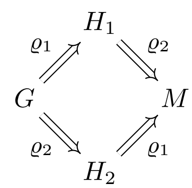
The following Local Church-Rosser Theorem generalises the corresponding result in [6] from $\mathcal{M}$-adhesive transformation systems to $\mathcal{M},\mathcal{N}$-adhesive transformation systems.

Theorem 2 (Local Church-Rosser Theorem).

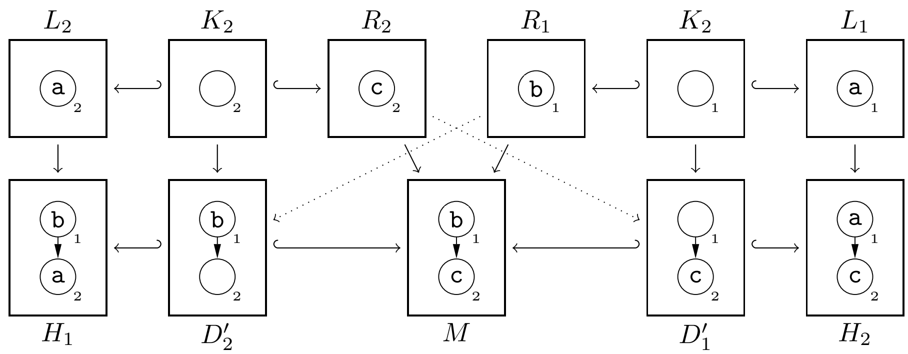
1. Given parallelly independent direct transformations $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2} H_2$, there are an object M and direct transformations $H_1 \Rightarrow_{\varrho_2,g_2'} M \Leftarrow_{\varrho_1,g_1'} H_2$

such that $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g'_2} M$ and $G \Rightarrow_{\varrho_2,g_2} H_2 \Rightarrow_{\varrho_1,g'_1} M$ are sequentially independent.

2. Given sequentially independent direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g_2} M$, there are an object H_2 and direct transformations $G \Rightarrow_{\varrho_2,g'_2} H_2 \Rightarrow_{\varrho_1,g'_1} M$ such that $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g'_2} H_2$ are parallelly independent:

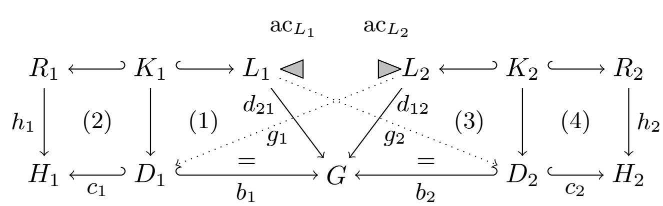


Example 4. Given the parallelly independent direct transformations in Example 4, there are a graph M and direct transformations $H_1 \Rightarrow_{\varrho_2} M \Leftarrow_{\varrho_1} H_2$ such that $G \Rightarrow_{\varrho_i} H_i \Rightarrow_{\varrho_j} M$ ($i \in \{1,2\}$ and $i \neq j$) are sequentially independent. Note that the direct transformations $G \Leftarrow_{\varrho_i^{-1}} H_i \Rightarrow_{\varrho_j} M$ ($i \in \{1,2\}$ and $i \neq j$) are parallelly independent.



We prove the Local Church-Rosser Theorem for $\mathcal{M},\mathcal{N}$-adhesive transformation systems by generalizing the corresponding proof for $\mathcal{M}$-adhesive systems in [4, 6].

Proof. 1. Let $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2} H_2$ be parallel independent. Then there are morphisms $d_{ij} \colon L_i \to D_j$ such that $g_i = (b_j \circ d_{ij})$, $g'_i = (c_j \circ d_{ij}) \models \mathrm{ac}_{L_i}$, and d'_i in $\mathcal{N}$ ($i, j \in \{1,2\}$ and $i \neq j$) (see the figure below).



The decomposition of the morphisms $g_i\colon L_i \to G$ into morphisms $d_{ij}\colon L_i \to D_j$ and $b_j\colon D_j \to G$ is used for a decomposition of the pushouts (i) into pushouts (i1) and (i2) for $i = 1, \ldots, 4$ (see Figure 1): Since $\mathcal{C}$ has $\mathcal{M}$-pullbacks, D_0 can be constructed as pullback object of $D_1 \hookrightarrow G \hookleftarrow D_2$. By the universal pullback property, there exist unique morphisms $K_i \to D_0$ $(i = 1, 2)$ such that (11) and (31) and the corresponding triangles commute. By the $\mathcal{M}, \mathcal{N}$ pushout-pullback decomposition property, the diagrams (11), (12), (31)=(12), and (32) are pushouts and pullbacks. Since $\mathcal{M}$ is stable under pushouts, the morphisms $D_i \hookrightarrow G$ are in $\mathcal{M}$. Since $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, $g_i = b_j \circ d_{ij}$ in $\mathcal{N}$, and $b_j\colon D_j \to G$ in $\mathcal{M}$, the morphisms $d_{ij}\colon L_i\colon D_j$ are in $\mathcal{N}$. Since $\mathcal{N}$ is stable under pullbacks, the morphisms $K_i \to D_0$ are in $\mathcal{N}$. Since $\mathcal{C}$ has $\mathcal{M}, \mathcal{N}$-pushouts, D'_j can be constructed as pushout object over $K_i \hookrightarrow R_i$ in $\mathcal{M}$ and $K_i \to D_0$ in $\mathcal{N}$. Since $\mathcal{M}, \mathcal{N}$-pushouts are pullbacks, (21) and (41) are pullbacks. By the universal pushout property, there exist unique morphisms $D'_j \hookrightarrow H_i$ such that the corresponding diagrams commute. By pushout composition, diagrams (22) and (42) are pushouts. By $\mathcal{M}, \mathcal{N}$-pullback decomposition, diagrams (22) and (42) are also pullbacks. Now, all diagrams in Figure 1 are pushouts and pullbacks and, since $\mathcal{M}$ and $\mathcal{N}$ are stable under pullbacks and pushouts, the vertical morphisms in the upper row are in $\mathcal{N}$ and the ones in the lower row are in $\mathcal{M}$.

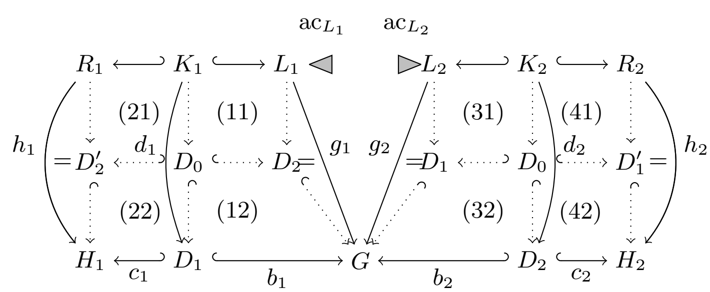


Fig. 1. Parallelly independent direct transformations

The pushouts can be rearranged as in Figure 2: By assumption, the matching morphisms g'_i $(i = 1, 2)$ are in $\mathcal{N}$. Since $\mathcal{N}$ is stable under pullbacks, (31)+(22), (11)+(42) are pullbacks, and g'_i are in $\mathcal{N}$, the morphisms $d'_i\colon K_i \to D'_i$ are in $\mathcal{N}$. Since $\mathcal{C}$ has $\mathcal{M}, \mathcal{N}$-pushouts, M can be constructed as pushout object of $R_2 \hookleftarrow K_2 \to D'_2$. Since $\mathcal{C}$ has $\mathcal{M}, \mathcal{N}$ pushouts and the morphisms $K_i \to D_0$ are in $\mathcal{N}$, $the objects$D'$_1.D'_2$ can be constructed as pushout objects in (41) and (21). By $\mathcal{M}, \mathcal{N}$-pullback decomposition, diagram (5) becomes pushout and pullback.

Define now the morphisms g'_i, d'_i, and h'_i $(i = 1, 2)$ as the composed morphisms in Figure 2. Then, the morphisms $d'_{12}\colon R_1 \to D'_2$ and $d'_{21}\colon L_2 \to D_2$ satisfy the commutativity requirements: $h_1 = (c'_2 \circ d'_{12})$ and $g'_2 = (b_1 \circ d'_{21})$. The morphisms d_i, d'_i are in $\mathcal{N}$ and, since $\mathcal{N}$ is stable under pushouts, the morphisms g'_i, h'_i are in $\mathcal{N}$. Finally, the requirements for the application conditions are satisfied: By

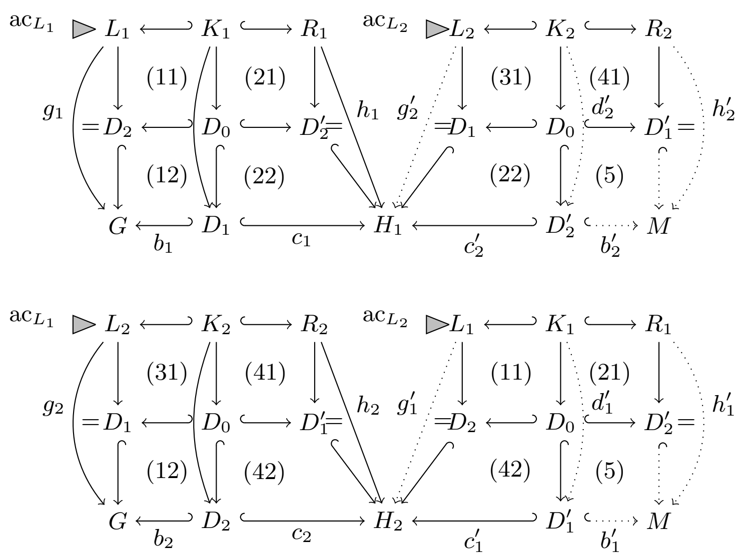


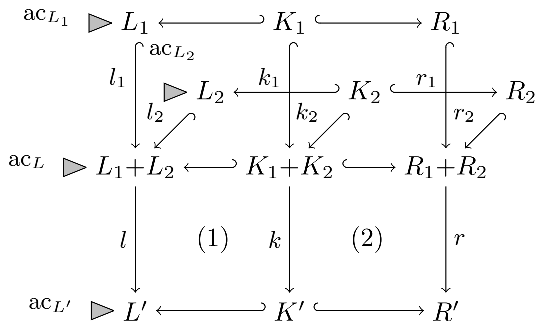
Fig. 2. Sequentially independent direct transformations

assumption, $g'_i \models \mathrm{ac}_{L_i}$ and, by the Shift Lemma 2, $g'_i \models \mathrm{ac}_{L_i}$ implies $h'_i = (b'_j \circ d'_{ij}) \models \mathrm{R}(\varrho_i, \mathrm{ac}_{L_i})$. $(i = 1, 2)$. Thus, the direct transformations $G \Rightarrow_{\varrho_i, g_i} H_i \Rightarrow_{\varrho_j, g'_j} M$ are sequentially independent.

2. The second statement can be proved with the help of the relationship between parallel and sequential independence and the first statement. □

Next we consider parallel rules, quotients rules, and parallel transformations. The parallel rule $\varrho_1 + \varrho_2$ of the rules ϱ_1 and ϱ_2 is defined by using the binary coproducts of the components of the rules. The quotient rule ϱ' of the rule ϱ is defined by using the $\mathcal{E}'$-$\mathcal{N}$ pair factorization of the rule. Both exist by the Assumption 1.

Definition 6 (Parallel rule, quotient rule, parallel transformation). Given rules ϱ_1 and ϱ_2, the *parallel rule* of ϱ_1 and ϱ_2 is the rule $\varrho_1 + \varrho_2 = \langle p, \mathrm{ac}_L \rangle$ where $p = \langle L_1 + L_2 \hookleftarrow K_1 + K_2 \hookrightarrow R_1 + R_2 \rangle$ and $\mathrm{ac}_L = \wedge_{i=1}^{2} \mathrm{Shift}(l_i, \mathrm{ac}_{L_i}) \wedge \mathrm{L}(p, \mathrm{Shift}(r_i, \mathrm{R}(\varrho_i, \mathrm{ac}_{L_i})))$.



The rule ϱ' above is a *quotient rule* of a rule $\varrho_1 + \varrho_2$ if there are two pushouts (1) and (2), where $l\colon L \to L' \in \mathcal{E}$, the morphisms $l \circ l_i$ are in $\mathcal{N}$ $(i = 1, 2)$, and $\mathrm{ac}_{L'} = \mathrm{Shift}(l, \mathrm{ac}_L)$. The set of quotient rules of a rule $\varrho_1 + \varrho_2$ is denoted by $\mathrm{Q}(\varrho_1 + \varrho_2)$. A direct transformation via a quotient of a parallel rule is called *parallel* direct transformation or parallel transformation, for short.

Fact 3 ([6]). $K_1{+}K_2 \hookrightarrow L_1{+}L_2$ and $K_1{+}K_2 \hookrightarrow R_1{+}R_2$ are in $\mathcal{M}$.

Example 5. The parallel rule of the rules ϱ_1 and ϱ_2 in Example 4 changing the label a into b and the label a into c, respectively, is the rule

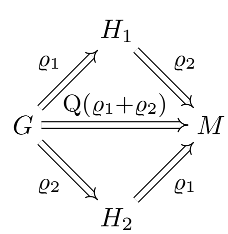
$$\varrho_1 + \varrho_2 = \langle\, (\mathtt{a})_1\ (\mathtt{a})_2 \ \hookleftarrow\ (\ )_1\ (\ )_2 \ \hookrightarrow\ (\mathtt{b})_1\ (\mathtt{c})_2 \,\rangle$$

changing the first label a into b and the second one into c.

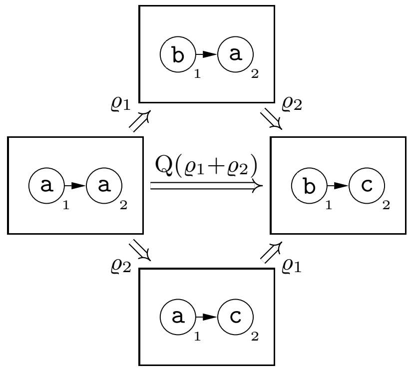
The connection between sequentially independent direct transformations and parallel direct transformations is given in the Parallelism Theorem.

Theorem 3 (Parallelism Theorem).

1. Synthesis. Given two sequentially independent direct transformations $G \Rightarrow_{\varrho_1, g_1} H_1 \Rightarrow_{\varrho_2, g'_2} M$, there is a parallel transformation $G \Rightarrow_{\mathrm{Q}(\varrho_1+\varrho_2), g} M$.
2. Analysis. Given a parallel transformation $G \Rightarrow_{\mathrm{Q}(\varrho_1+\varrho_2), m} M$, there are sequentially independent direct transformations $G \Rightarrow_{\varrho_1, g_1} H_1 \Rightarrow_{\varrho_2, g'_2} M$ and $G \Rightarrow_{\varrho_2, g_2} H_2 \Rightarrow_{\varrho_1, g'_1} M$.
3. Bijective correspondence. The synthesis and analysis constructions are inverse to each other up to isomorphism:

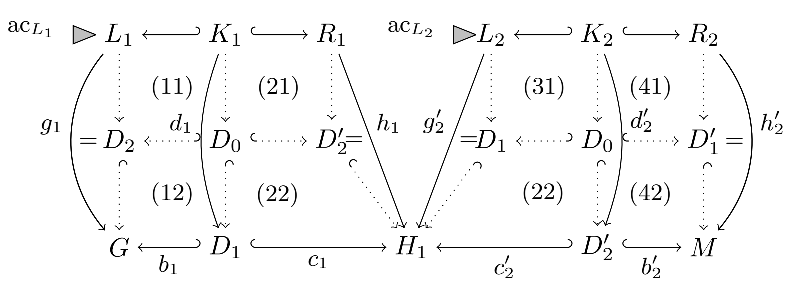


Example 6. Given two sequentially independent direct transformations as in the upper part of the figure, there is a parallel transformation in the middle of the figure. Given a parallel transformation in the middle of the figure, there are sequentially independent direct transformations in the upper and the lower part of the figure.

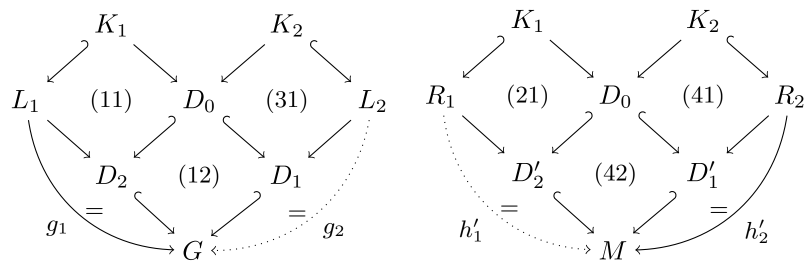


We lift the proof of the Parallelism Theorem for $\mathcal{M}$-adhesive transformation systems in [4, 6] to the setting of $\mathcal{M},\mathcal{N}$-adhesive systems.

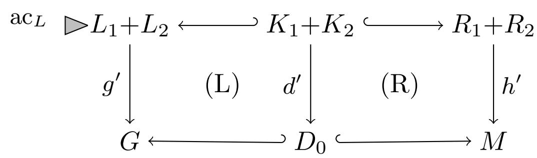
Proof. 1. Given two sequentially independent direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g'_2} M$. By $\mathcal{M}$-pullback construction, there is a decomposition of the pushouts into pushouts (i1) and (i2) for $i = 1, \ldots, 4$ as shown below. The pushouts are also pullbacks. Since $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, the morphisms in the upper row are in $\mathcal{N}$.



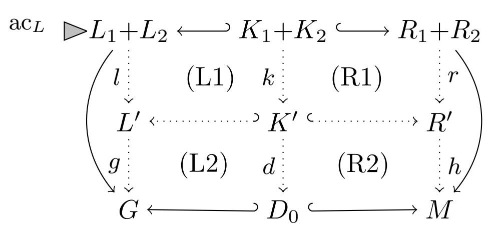
The pushouts can be composed as butterflies as shown below:



By the Butterfly Lemma [4], the folded butterflies (L) and (R) are pushouts.

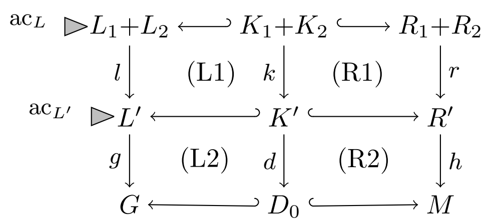


In general, the morphisms g', d', h' are not in $\mathcal{N}$. Therefore, we construct a quotient rule of $\varrho_1+\varrho_2$: By $\mathcal{E}$-$\mathcal{N}$ factorization (which exists by the Assumption 1) of the morphism g' in $\mathcal{N}$, there exist an object L' and morphisms $l\colon L_1 + L_2 \to L'$, $g\colon L' \to G$ such that $g' = g \circ l$, $l \in \mathcal{E}$, and $g \in \mathcal{N}$. Construct now a pullback object K' of $L' \to G \hookleftarrow D_0$ (L2). By universal pullback property, there is some morphism $K_1 + K_2 \to K'$ such that diagram (L1) and the triangle commute. By $\mathcal{M},\mathcal{N}$-pushout-pullback decomposition, diagrams (L1) and (L2) are pushouts and pullbacks. Since $\mathcal{C}$ has $\mathcal{M},\mathcal{E}$-pushouts, we can construct R' as pushout object of $K' \leftarrow K_1 + K_2 \hookrightarrow R_1 + R_2$ (R1). By universal pushout property, there is some morphism $R' \to M$ such that (R2) commutes and the triangle commutes. By pushout decomposition, (R2) is a pushout. Since $\mathcal{M},\mathcal{E}$-pushouts are pullbacks, (R1) is also a pullback. Thus, (L1), (L2), (R1), (R2) are pushouts. Since $\mathcal{N}$ is closed under decomposition, the morphisms $l'_i = l \circ l_i$ are in $\mathcal{N}$. Thus, $\varrho' = \langle p', \mathrm{ac}_{L'}\rangle$ with plain rule $p' = \langle L' \hookleftarrow K' \hookrightarrow R'\rangle$ and left application condition $\mathrm{ac}_{L'} = \mathrm{Shift}(l, \mathrm{ac}_L)$ is a quotient rule of $\varrho_1 + \varrho_2$.

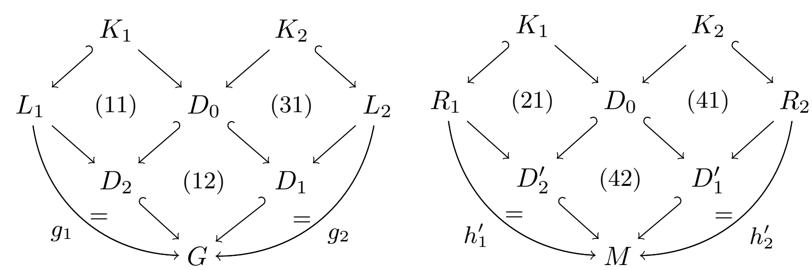


Since $\mathcal{N}$ is stable under pushouts, the morphisms g, h are in $\mathcal{N}$. Moreover, we have $g \models \mathrm{ac}_{L'}$ because $g_1 \models \mathrm{ac}_{L_1}$ and $g'_2 \models \mathrm{ac}_{L_2}$ and $g_1 \models \mathrm{ac}_{L_1}$ and $g'_2 \models \mathrm{ac}_{L_2}$ implies $g \models \mathrm{ac}_{L'}$ [6]. Consequently, $G \Rightarrow_{\varrho',g} M$ is a parallel transformation.

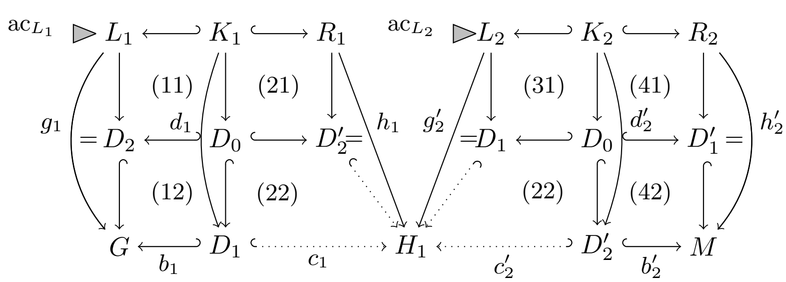
2. Given a parallel transformation $G \Rightarrow_{\mathrm{Q}(\varrho_1+\varrho_2),g} M$, then there is a quotient rule ϱ' of $\varrho_1+\varrho_2$ such that $G \Rightarrow_{\varrho',g} M$. Thus there are pushouts (L1) and (R1), where $l\colon l_1+L_2 \to L'$ in $\mathcal{E}$, $l \circ l_i$ in $\mathcal{N}$ $(i=1,2)$, and $g\colon L' \to G$ in $\mathcal{N}$, and $\mathrm{ac}_{L'} = \mathrm{Shift}(l, \mathrm{ac}_L)$. Since $\mathcal{M},\mathcal{N}\cup\mathcal{E}$-pushouts are pullbacks, all diagrams below are pullbacks. By pushout and plullback composition, the composed diagrams (L1)+(L2) and (R1)+(R2) are pushouts and pullbacks.



By the Butterfly Lemma [4], there exist decompositions of the composed pushouts into pushouts as given below. By $\mathcal{M},\mathcal{N}$-pullback decomposition, the all diagrams are pullbacks. Since $\mathcal{N}$ is closed under composition, the morphism $g_1 = g \circ l \circ l_1$ is in $\mathcal{N}$. Since $\mathcal{N}$ is stable under pullbacks, the morphism d_1 is in $\mathcal{N}$. Similarly, the morphism d_2' is in $\mathcal{N}$.



The diagrams can be composed in a new fashion.



Since $\mathcal{C}$ has $\mathcal{M},\mathcal{N}$-pushouts, H_1 can be constructed as pushout object in $R_1 \hookleftarrow K_1 \to D_1$. By the universal pushout property, there is some $D_2' \to H_1$ such that diagram (22) commutes. By pushout decomposition, (22) is a pushout.

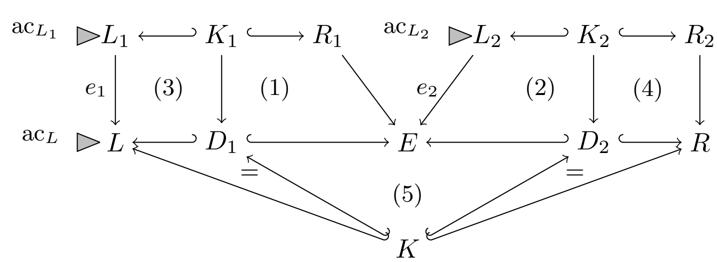
Now, all diagrams in the figure above are pushouts. By pushout composition, all composed diagrams are pushouts. Since $\mathcal{N}$ is stable under pushouts and d_1, d'_2 are in $\mathcal{N}$, all composed vertical morphisms are in $\mathcal{N}$. Since, by assumption, $g \models \mathrm{ac}_L$ and $g \models \mathrm{ac}_L$ iff $g_1 \models \mathrm{ac}_{L_1}$ and $g'_2 \models \mathrm{ac}_{L_2}$ [6], we have $g_1 \models \mathrm{ac}_{L_1}$ and $g'_2 \models \mathrm{ac}_{L_2}$. Moreover, there are morphisms $d_{12}\colon R_1 \to D'_2$ and $d_{21}\colon L_2 \to D_1$ such that $h_1 = c'_2 \circ d_{12}$ and $g'_2 = c_1 \circ d_{21}$. Thus, the direct transformations $G \Rightarrow_{\varrho_1, g_1} H_1 \Rightarrow_{\varrho_2, g'_2} M$ are sequentially independent. Accordingly, one may construct sequentially independent direct transformations $G \Rightarrow_{\varrho_2, g_2} H_2 \Rightarrow_{\varrho_1, g'_1} M$.

3. Because of the uniqueness of pushouts and pushout complements, the above constructions are inverse to each other up to isomorphism. □

Finally, we consider transformations $G \Rightarrow_{\varrho_1} H \Rightarrow_{\varrho_2} M$ that need not be sequentially independent. This leads to the notions of an E-dependency relation, an E-concurrent rule for ϱ_1 and ϱ_2, E-concurrent transformations, and E-related transformations. The connection between E-related and E-concurrent transformations is established in the Concurrency Theorem.

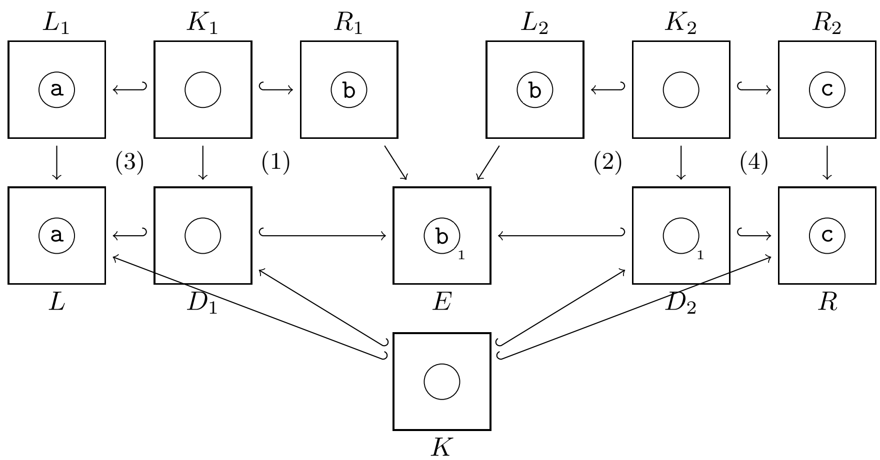
The construction of an E-concurrent rule is based on an E-dependency relation which guarantees the existence of some pushout complements. It is defined using $\mathcal{M},\mathcal{N}$-pushouts and $\mathcal{M}$-pullbacks. The application condition of the E-concurrent rule guarantees that, whenever the E-concurrent rule is applicable, the rules ϱ_1 and ϱ_2 are applicable in sequential order.

Definition 7 (E-concurrent rule). Given two rules ϱ_1 and ϱ_2, an object E with morphisms $e_1^*\colon R_1 \to E$ and $e_2\colon L_2 \to E$ is an *E-dependency relation* for ϱ_1 and ϱ_2 if (e_1^*, e_2) in $\mathcal{E}'$ and the pushout complements (1) and (2) over $K_1 \hookrightarrow R_1 \to E$ and $K_2 \hookrightarrow L_2 \to E$ exist such that $K_1 \to D_1$ and $K_2 \to D_2$ are in $\mathcal{N}$. Given such an E-dependency relation for ϱ_1 and ϱ_2, the *E-concurrent rule* of ϱ_1 and ϱ_2 is the rule $\varrho_1 *_E \varrho_2 = \langle p, \mathrm{ac}_L\rangle$ where $p = \langle L \hookleftarrow K \hookrightarrow R\rangle$ with pushouts (3), (4) and pullback (5), $\varrho_1^* = \langle L \hookleftarrow D_1 \hookrightarrow E\rangle$, and $\mathrm{ac}_L = \mathrm{Shift}(e_1, \mathrm{ac}_{L_1}) \wedge \mathrm{L}(\varrho_1^*, \mathrm{Shift}(e_2, \mathrm{ac}_{L_2}))$.

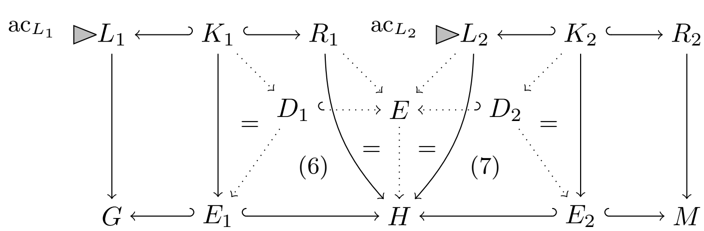


Remark 5. Since, by Theorem 1, pushout complements are unique (up to isomorphisms) and $\mathcal{C}$ has $\mathcal{M},\mathcal{N}$-pushouts and $\mathcal{M}$-pullbacks, the construction of the E-concurrent rule is unique (up to isomorphism). Since $\mathcal{M}$ is closed under composition, the morphisms in the rule are in $\mathcal{M}$ and, since $\mathcal{N}$ is stable under pushouts, $K_i \to D_i$ in $\mathcal{N}$ $(i = 1, 2)$ implies that all vertical morphisms are in $\mathcal{N}$.

Example 7. Consider the rules ϱ_1 and ϱ_2 below. Then the graph $E =$ (b) together with the identities $\mathrm{id}_E, \mathrm{id}_E$ is an E-dependency relation for ϱ_1 and ϱ_2. The E-concurrent rule $\varrho_1 *_E \varrho_2$ is depicted in the figure below. Note that the directed transformations are not sequentially independent.



Definition 8 (E-concurrent and E-related transformation). A direct transformation via an E-concurrent rule is called E-*concurrent* direct transformation or E-concurrent transformation, for short. A transformation $G \Rightarrow_{\varrho_1} H \Rightarrow_{\varrho_2} M$ is E-*related* if there are morphisms $E \to H$, $D_i \to E_i$, $K \to F$ are in $\mathcal{N}$ $(i = 1, 2)$ such that the diagrams (6) and (7) are pushouts and the triangles commute.

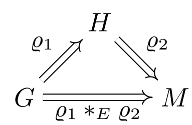


where K and F are the pullback objects over $D_1 \hookrightarrow E \hookleftarrow D_2$ and $E_1 \hookrightarrow H \hookleftarrow E_2$, respectively, and $K \to F$ is the uniquely exists morphism.

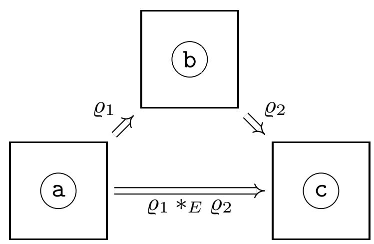
Theorem 4 (Concurrency Theorem). Let E be a dependency relation for rules ϱ_1 and ϱ_2.

1. Synthesis. Given an E-related transformation $G \Rightarrow_{\varrho_1, g_1} H \Rightarrow_{\varrho_2, g_2} M$, there is an E-concurrent transformation $G \Rightarrow_{\varrho_1 *_E \varrho_2, g} M$.
2. Analysis. Given an E-concurrent transformation $G \Rightarrow_{\varrho_1 *_E \varrho_2, g} M$, there is an E-related transformation $G \Rightarrow_{\varrho_1, g_1} H \Rightarrow_{\varrho_2, g_2} M$.

3. Bijective correspondence. The synthesis and analysis constructions are inverse to each other up to isomorphism, provided that every pair (e_1, e_2) in $\mathcal{E}'$ is jointly epimorphic[4]:

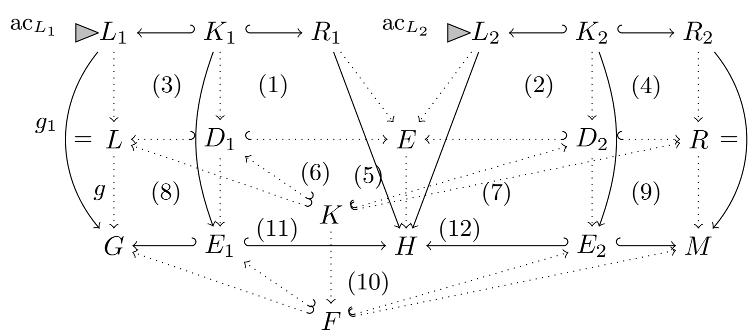


Example 8. Consider the E-dependency relation for ϱ_1 and ϱ_2 in Example 7. Given the E-related transformation in the figure below, there is an E-concurrent transformation. Given an E-concurrent transformation in the figure below, there is an E-related transformation.



We now generalize the proof of the Concurrency Theorem for $\mathcal{M}$-adhesive transformation systems in [4, 6] to $\mathcal{M}, \mathcal{N}$-adhesive systems.

Proof. 1. Consider the E-related transformation $G \Rightarrow_{\varrho_1, g_1} H \Rightarrow_{\varrho_2, g_2} M$ below. By Definitions 7 and 8, diagrams (1)-(4) below are pushouts, (5) is a pullback, and there are morphisms in $\mathcal{N}$ such that (6) and (7) are pushouts and the triangles commute. By the universal pushout property, there are unique morphisms $L \rightarrow G$ and $R \rightarrow M$, respectively, such that the corresponding diagrams commute. By pushout decomposition, diagrams (8) and (9) are pushouts.



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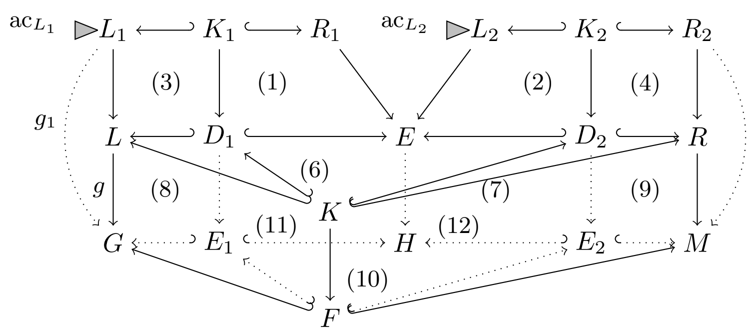
[4] A morphism pair (e_1, e_2) with $e_i \colon A_i \rightarrow B$ $(i = 1, 2)$ is *jointly epimorphic*, if for all $g, h \colon B \rightarrow C$ with $g \circ e_i = h \circ e_i$ for $i = 1, 2$, we have $g = h$.

Since $\mathcal{C}$ has $\mathcal{M}$-pullbacks, we can construct the pullback (10). By the universal pullback property, there is a unique morphism $K \to F$ such that the corresponding diagrams commute. By the cube $\mathcal{M}, \mathcal{N}$ pushout-pullback decomposition property to the cube in the diagram, where the top and bottom are pullbacks and the back faces are pushouts, the front faces (11) and (12) are also pushouts. By pushout composition, the diagrams (11)+(8) and (12)+(9) are pushouts. By Definition 8, the morphisms $E \to H$, $D_i \to E_i$, $K \to F$ are in $\mathcal{N}$ $(i = 1, 2)$. By stability of $\mathcal{N}$ under pushouts, all vertical morphisms in the lower row of the figure above are in $\mathcal{N}$. Moreover, $g_i \models \mathrm{ac}_{L_i}$ for $i = 1, 2$ implies $g \models \mathrm{ac}_L$ [6]. Thus, there is an E-concurrent transformation $G \Rightarrow_{\varrho_1 *_E \varrho_2, g} M$.

2. Consider the E-concurrent transformation $G \Rightarrow_{\varrho, g} M$ via $\varrho = \varrho_1 *_E \varrho_2$.



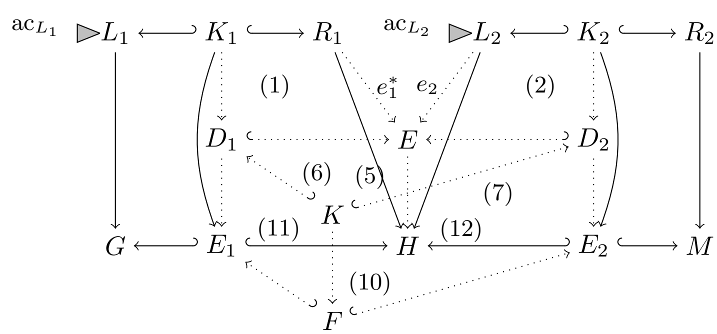
Since $\mathcal{N}$ is stable under pushouts and pullbacks, the morphisms $K \to D_i, D_i \to E$ are in $\mathcal{N}$. By assumption, $K \to F$ is in $\mathcal{N}$. Since $\mathcal{C}$ has $\mathcal{M}, \mathcal{N}$-pushouts, $K \hookrightarrow D_i$ in $\mathcal{M}$, and $K \to F$ in $\mathcal{N}$, we can construct the pushout (1i) over $D_i \hookleftarrow K \to F$, leading to the objects E_i $(i = 1, 2)$. By the universal pushout property, there are unique morphism $E_1 \hookrightarrow G$ and $E_2 \hookrightarrow M$ such that diagrams (8) and (9) and the corresponding triangles commute. Since $\mathcal{C}$ has $\mathcal{M}, \mathcal{N}$-pushouts, $D_1 \hookrightarrow E$ in $\mathcal{M}$, and $D_1 \to E_1$ in $\mathcal{N}$, we can construct the pushout (6) over $E \hookleftarrow D_1 \to E_1$, leading to the object H. By the universal pushout property, there is a unique morphism $E_2 \hookrightarrow H$ such that diagram (7) and the corresponding diagrams commute. By pushout composition and decomposition applied to the resulting cube, diagram (7) is also a pushout. Since $\mathcal{N}$ is stable under pushouts and, by Definitions 7 and 8, $K_i \to D_i, D_i \to E_i, K \to F$ in $\mathcal{N}$, all vertical morphisms in the figure above are in $\mathcal{N}$. Since $\mathcal{N}$ is closed under composition, all composed vertical morphisms are in $\mathcal{N}$. Since, by assumption, $g \models \mathrm{ac}_L$ and $g \models \mathrm{ac}_L$ iff $g_i \models \mathrm{ac}_{L_i}$ for $i = 1, 2$ [6], we have $g_i \models \mathrm{ac}_{L_i}$ for $i = 1, 2$. Thus, there are E-related direct transformations $G \Rightarrow_{\varrho_1, g_1} H \Rightarrow_{\varrho_2, g_2} M$.

3. The bijective correspondence follows from the fact that pushouts and pullback constructions are unique upt o isomorphism and that the pair $(e_1, e_2) \in \mathcal{E}'$ is jointly epimorphic, leading to a unique morphism $E \to H$ in Definition 8. $\square$

In order to apply the Concurrency Theorem to a transformation, it remains to construct an E-related transformation. The fact below makes use of $\mathcal{E}'$-$\mathcal{M} \cap \mathcal{N}$ pair factorization.

Fact 4 (Construction of E-related transformations). Let $\mathcal{C}$ have an $\mathcal{E}', \mathcal{M} \cap \mathcal{N}$ pair factorization. For every transformation $G \Rightarrow_{\varrho_1,g_1} H \Rightarrow_{\varrho_2,g_2} M$, an E-dependency relation E can be constructed such that $G \Rightarrow_{\varrho_1,g_1} H \Rightarrow_{\varrho_2,g_2} M$ is E-related.

Proof. The construction is similar to that in Fact 5.29 in [4]. Given a transformation $G \Rightarrow_{\varrho_1,g_1,h_1} H \Rightarrow_{\varrho_2,g_2} M$. By $\mathcal{E}'$-$\mathcal{M} \cap \mathcal{N}$ pair factorization, there are an object E and morphisms $e_1^*\colon R_1 \to E$ and $e_2\colon L_2 \to E$ with $(e_1^*, e_2) \in \mathcal{E}'$ and $h\colon E \hookrightarrow H$ in $\mathcal{M} \cap \mathcal{N}$ such that $h \circ e_1^* = h_1$ and $h \circ e_2 = g_2$. Since $\mathcal{C}$ has $\mathcal{M}$-pullbacks, we can construct diagrams (6) and (7) as pullbacks of $E_i \hookrightarrow H \hookleftarrow E$. By the universal pullback property, there are morphisms $K_i \to D_i$ $(i = 1, 2)$ such that diagram (1) and (2) and the corresponding triangles commute. By the $\mathcal{M}, \mathcal{N}$-pushout-pullback decomposition property, (1)+(6) pushout, (6) pullback, and h in $\mathcal{M}$, the diagrams (1) and (6), and according, (2) and (7) are pushouts.



Since $\mathcal{N}$ is stable under pullbacks and h in $\mathcal{N}$, the morphisms $D_i \to E_i$ are in $\mathcal{N}$. Since $\mathcal{N}$ is closed under composition and $K_i \to D_i, D_i \to E_i$ in $\mathcal{N}$, the composed morphisms $K_i \to E_i = K_i \to D_i \to E_i$ are in $\mathcal{N}$ and, by $D_i \to E_i$ in $\mathcal{N}$, the morphisms $K_i \to D_i$ are in $\mathcal{N}$. It remains to show that $K \to F$ is in $\mathcal{N}$. Since $\mathcal{C}$ has $\mathcal{M}$-pullbacks, we can construct the pullbacks (5) and (10). By the universal pullback property, there is a unique morphism $K \to F$ such that the corresponding diagrams commute. By the cube $\mathcal{M}, \mathcal{N}$ pushout-pullback decomposition property to the cube in the diagram, where the top and bottom are pullbacks and the back faces are pushouts, the front faces (11) and (12) are also pushouts. By pullback decomposition, (5)+(7), (10) pullbacks implies that diagram (11) is a pullback. Since $\mathcal{N}$ is stable under pullbacks, (11) is a pullback, and $D_i \to E_i$ in $\mathcal{N}$, the morphism $K \to F$ is in $\mathcal{N}$. Thus, E is an E-dependency relation for ϱ_1 and ϱ_2 and $G \Rightarrow_{\varrho_1,g_1} H \Rightarrow_{\varrho_2,g_2} M$ is E-related. □

4 Category PLG is $\mathcal{M},\mathcal{N}$-adhesive

In this section, we consider the category PLG of partially labelled graphs [15]. We first show that PLG is not $\mathcal{M}$-adhesive for the class $\mathcal{M}$ of injective graph morphisms. We then prove that PLG is $\mathcal{M},\mathcal{N}$-adhesive, though, and satisfies the HLR$^+$-properties if we choose $\mathcal{N}$ as a suitable class of morphisms. As a consequence, we obtain the Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem as new results for the setting of graph transformation with relabelling and application conditions.

We start by recalling the basic notions of partially labelled graphs and their morphisms.

Definition 9 (Graphs and morphisms). A *(partially labelled) graph* is a system $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G, \mathrm{l}_{G,\mathrm{V}}, \mathrm{l}_{G,\mathrm{E}})$ consisting of finite sets V_G and E_G of *nodes* and *edges*, source and target functions $\mathrm{s}_G, \mathrm{t}_G\colon \mathrm{E}_G \to \mathrm{V}_G$, and partial labelling functions $\mathrm{l}_{G,\mathrm{V}}\colon \mathrm{V}_G \to \mathrm{C}_\mathrm{V}$ and $\mathrm{l}_{G,\mathrm{E}}\colon \mathrm{E}_G \to \mathrm{C}_\mathrm{E}$,[5] where C_V and C_E are fixed sets of node and edge labels. A graph G is *totally labelled* if $\mathrm{l}_{G,\mathrm{V}}$ and $\mathrm{l}_{G,\mathrm{E}}$ are total functions.

A *morphism* $g\colon G \to H$ between graphs G and H consists of two functions $g_\mathrm{V}\colon \mathrm{V}_G \to \mathrm{V}_H$ and $g_\mathrm{E}\colon \mathrm{E}_G \to \mathrm{E}_H$ that preserve sources, targets and labels, that is, $\mathrm{s}_H \circ g_\mathrm{E} = g_\mathrm{V} \circ \mathrm{s}_G$, $\mathrm{t}_H \circ g_\mathrm{E} = g_\mathrm{V} \circ \mathrm{t}_G$, and $\mathrm{l}_H(g(x)) = \mathrm{l}_G(x)$ for all x in $\mathrm{Dom}(\mathrm{l}_G)$[6]. Such a morphism *preserves undefinedness* if it maps unlabelled items in G to unlabelled items in H. Moreover, it *strongly preserves undefinedness* if, in addition, it is injective on unlabelled items. Morphism g is *injective* (*surjective*) if g_V and g_E are injective (surjective), and an *isomorphism* if it is injective, surjective and preserves undefinedness. In the latter case G and H are *isomorphic*, which is denoted by $G \cong H$. Furthermore, g is an *inclusion* if $g(x) = x$ for all x in G (note that inclusions need not preserve undefinedness). The *composition* $h \circ g$ of g with a morphism $h\colon H \to M$ consists of the composed functions $h_\mathrm{V} \circ g_\mathrm{V}$ and $h_\mathrm{E} \circ g_\mathrm{E}$. We write PLG for the category having partially labelled graphs as objects and graph morphisms as arrows.

In pictures of graphs, nodes are drawn as circles with their labels (if existent) inside, and edges are drawn as arrows with their labels (if existent) placed next to them. Graph morphisms are graphically represented by attaching the same number to nodes and their images.

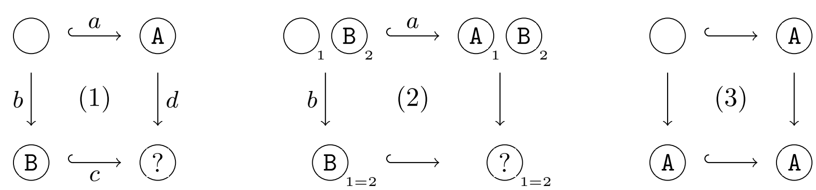
While the category of labelled graphs with arbitrary morphisms has pushouts [3], the category of partially labelled graphs with injective morphisms has no pushouts [15]. As a consequence, the category PLG with the class $\mathcal{M}$ of injective morphisms is not $\mathcal{M}$-adhesive.

[5] Given sets A and B, a partial function $f\colon A \to B$ is a function from some subset A' of A to B. The set A' is the *domain* of f and is denoted by $\mathrm{Dom}(f)$. We say that $f(x)$ is *undefined*, and write $f(x) = \bot$, if x is in $A - \mathrm{Dom}(f)$.

[6] We often do not distinguish between nodes and edges in statements that hold analogously for both sets.

Fact 5 (PLG **is not** $\mathcal{M}$**-adhesive).** Let $\mathcal{M}$ be the class of injective graph morphisms. Then PLG does not have $\mathcal{M}$-pushouts. Moreover, $\mathcal{M}$-pushouts need not be pullbacks.

Example 9. In square (1), the morphism a is injective, the morphism b is not undefinedness preserving, and a pushout does not exist: it is impossible to make both morphisms c and d label preserving. In square (2), a is injective, b is undefinedness preserving, but not sstrongly undefinedness preserving, and a pushout does not exist. Square (3) is an $\mathcal{M}$-pushout, but not a pullback.



Assumption 3. For the rest of this section, we consider the category PLG and let $\mathcal{M}$ be the class of injective graph morphisms and $\mathcal{N}$ the class of strongly undefinedness preserving graph morphisms.

Theorem 5. The category PLG is $\mathcal{M},\mathcal{N}$-adhesive.

To prove Theorem 5, we establish the properties required by Definition 1 in the following five lemmata.

Lemma 3 (Closure properties). $\mathcal{M}$ and $\mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition. Moreover, $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition.

Proof. Straightforward. □

Lemma 4 (PLG **has** $\mathcal{M},\mathcal{N}$**-pushouts).** Given graph morphisms $r\colon K \hookrightarrow R$ in $\mathcal{M}$ and $d\colon K \to D$ in $\mathcal{N}$, there exist a graph H and graph morphisms $c\colon D \hookrightarrow H$ and $h\colon R \to H$ such that square (2) below is a pushout.

$$\begin{array}{ccc} K & \overset{r}{\hookrightarrow} & R \\ d\downarrow & (2) & \downarrow h \\ D & \underset{c}{\hookrightarrow} & H \end{array}$$

Construction. The sets of nodes and edges are defined by $H = (D - d(K)) + R$. The source function s_H is defined by $\mathrm{s}_H(e) = $ if $e \in E_R$ then $\mathrm{s}_R(e)$ else $\mathrm{s}_D(e)$;

the target function t_D is defined analogously. The labelling functions l_H are defined by

$$l_H(x) = \begin{cases} l_R(x) & \text{if } x \in R \text{ and } l_R(x) \neq \perp, \\ l_D(d(x')) & \text{if } x \in R,\, l_R(x) = \perp,\, r(x') = x, \\ l_D(x) & \text{if } x \in (D - d(K)). \end{cases}$$

Morphism $h\colon R \to H$ is the inclusion of R in H and $c\colon D \hookrightarrow H$ is defined by $c(x) = $ if $x \in D - d(K)$ then x else $r(k)$ for the unique $k \in K$ with $d(k) = x$.

Proof. See [15]. □

The category PLG has not only $\mathcal{M}$-pullbacks but possesses all pullbacks.

Lemma 5 (PLG **has pullbacks).** Let $c\colon D \to H$ and $h\colon R \to H$ be graph morphisms. Then there exist a graph K and graph morphisms $d\colon K \to D$ and $r\colon K \to R$ such that square (2) above is a pullback.

Construction. The sets of nodes and edges are defined by

$$K = \{\langle x, y\rangle \in D \times R \mid c(x) = h(y)\}.$$

The source function s_K is defined by $s_K(\langle x, y\rangle) = \langle s_D(x), s_R(y)\rangle$, the target function t_K is defined analogously. The labelling functions l_K are defined by

$$l_K(\langle x, y\rangle) = \text{ if } (l_D(x) = l_R(y) \neq \perp) \text{ then } l_R(x) \text{ else } \perp.$$

The morphisms $d\colon K \to D$ and $r\colon K \to R$ are the projections from $D \times R$ to D and R, that is, they are given by $d(\langle x, y\rangle) = x$ and $r(\langle x, y\rangle) = y$.

Proof. See [15]. □

Lemma 6 ($\mathcal{M}$ and $\mathcal{N}$ are stable). The classes $\mathcal{M}$ and $\mathcal{N}$ are stable under pushouts and pullbacks.

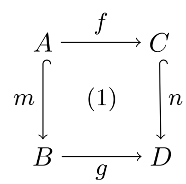
Proof. This follows from the construction of $\mathcal{M},\mathcal{N}$-pushouts and pullbacks in Lemma 4 and Lemma 5, and the fact that pushouts and pullbacks are unique up to isomorphism. □

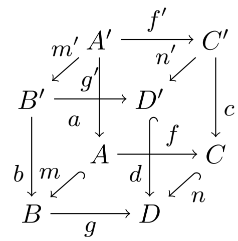
Lemma 7 ($\mathcal{M},\mathcal{N}$-van Kampen squares). $\mathcal{M},\mathcal{N}$-Pushouts are $\mathcal{M},\mathcal{N}$-van Kampen squares.

Proof. We exploit the fact that the category ULG of unlabelled graphs is $\mathcal{M}$-adhesive. (This follows from Fact 4.1.6 for labelled graphs in [4], by restricting the label alphabet to a single label.)

Consider the pushout (1) below where $m \in \mathcal{M}$ and $f \in \mathcal{N}$. We have to show that, given a commutative cube (2) with (1) as bottom face, $b, c, d \in \mathcal{M}$, and pullbacks as back faces, the following holds:

the top face is a pushout $\Leftrightarrow$ the front faces are pullbacks.





(2)

Part 1 ("$\Rightarrow$ "). Assume that the top face of cube (2) is a pushout. Since pullback objects are unique up to isomorphism, it is sufficient to prove that B' and C' are isomorphic to the corresponding pullback objects. Let B'' be the pullback object of g and d with morphisms $b''\colon B'' \to B$ and $g''\colon B'' \to D'$. By the universal property of pullbacks, there is a unique morphism $u\colon B' \to B''$ such that $b'' \circ u = b$ and $g'' \circ u = g$. By forgetting all labels, cube (2) becomes a cube in ULG. Since ULG is $\mathcal{M}$-adhesive, every pushout in ULG is a van Kampen square. Consequently, the morphism u is injective and surjective. It remains to show that u is $\perp$-preserving. Let $x \in B' - \mathrm{Dom}(\mathrm{l}_{B'})$. Suppose that $u(x) \in \mathrm{Dom}(\mathrm{l}_{B''})$. Then $b(x) \in \mathrm{Dom}(\mathrm{l}_B)$ and $g'(x) \in \mathrm{Dom}(\mathrm{l}_{D'})$.

Since the top is a pushout in PLG and $x \in B' - \mathrm{Dom}(\mathrm{l}_{B'})$, there exists $y \in \mathrm{Dom}(\mathrm{l}_{C'})$ with $g'(x) = n'(y)$. Since the bottom is a pushout in PLG and $m \in \mathcal{M}$, by Theorem 1, it is also a pullback, $b(x) \in \mathrm{Dom}(\mathrm{l}_B)$, $c(y) \in \mathrm{Dom}(\mathrm{l}_C)$, and the left front face commutes, $g(b(x)) = d(g'(x)) = d(n'(y))$ and there exists $z \in \mathrm{Dom}(\mathrm{l}_A)$ such that $m(z) = b(x)$ and $f(z) = c(y)$. Since the back right face is a pullback, $y \in \mathrm{Dom}(\mathrm{l}_{C'})$ and $z \in \mathrm{Dom}(\mathrm{l}_A)$ with $c(y) = m(z)$, there is some $x' \in \mathrm{Dom}(\mathrm{l}_{A'})$ with $m'(x') = x$. Then $x \in \mathrm{Dom}(\mathrm{l}_{B'})$, a contradiction. Thus u is $\perp$-preserving and B' and B'' are isomorphic. Similarly, it is shown that C' and the pullback object C'' of d and n are isomorphic. Thus, the back faces of cube (2) are pullbacks.

Part 2 ("$\Leftarrow$ ") Assume that the front faces of cube (2) are pullbacks in PLG. Since pushout objects are unique up to isomorphism, it is sufficient to prove that D' is isomorphic to the corresponding pushout object. Let D'' be the pushout object of m' and f' in PLG with morphisms $g''\colon B' \to D''$ and $n''\colon C' \to D''$. By the universal property of pushouts, there is a unique morphism $u\colon D'' \to D'$ such that $g' = u \circ g''$ and $n' = u \circ n''$. Consider now the underlying pushout in ULG. Since ULG is $\mathcal{M}$-adhesive, every pushout in ULG is a van Kampen square. Consequently, the morphism u is injective and surjective. It remains to show that u is $\perp$-preserving. Let $x \in D'' - \mathrm{Dom}(\mathrm{l}_{D''})$. Suppose that $u(x) \in \mathrm{Dom}(\mathrm{l}_{D'})$. Then $d(u(x)) \in \mathrm{Dom}(\mathrm{l}_D)$. Since the bottom is a pushout, there are two cases. In the first case, there exists an item $y \in \mathrm{Dom}(\mathrm{l}_B)$ such that $g(y) = d(u(x))$.

Since the left front face is a pullback, $y \in \mathrm{Dom}(\mathrm{l}_B)$ and $u(x) \in \mathrm{Dom}(\mathrm{l}_{D'})$ with $g(y) = d(u(x))$, there is some $z \in \mathrm{Dom}(\mathrm{l}_{B'})$ with $b(z) = y$ and $g'(z) = u(x)$. By commutativity of the left front face, $d(g'(z)) = g(b(z)) = g(y) = d(u(x))$. By $d \in \mathcal{M}$, $g'(z) = u(x) \in \mathrm{Dom}(\mathrm{l}_{D''})$, a contradiction. In the second case, there exists an item $y \in \mathrm{Dom}(\mathrm{l}_C)$ such that $n(y) = d(u(x))$. Since the right front face is a pullback, we obtain a contradiction. Thus, the morphism u is $\perp$-preserving and the top face is a pushout. Since the back faces are pullbacks and $\mathcal{M}$ and $\mathcal{N}$ are stable under $\mathcal{M}$-pullbacks, $m \in \mathcal{M}$ and $f \in \mathcal{N}$ imply $m' \in \mathcal{M}$ and $f' \in \mathcal{N}$, i.e. the top face is an $\mathcal{M},\mathcal{N}$-pushout. □

Proof of Theorem 5. See Lemma 3 to Lemma 7. □

Lemma 8 (HLR$^+$-properties). PLG satisfies the HLR$^+$-properties, where $\mathcal{E}$ is the class of surjective, strongly undefinedness preserving morphisms and $\mathcal{E}'$ is the class of pairs of jointly surjective, strongly undefinedness preserving morphisms.

Proof. Routine.

By Theorem 5 and Lemma 8, we obtain the following corollary.

Corollary 1. The Local Church-Rosser Theorem, the Parallelism Theorem, and the Concurrency Theorem hold for $\mathcal{M},\mathcal{N}$-adhesive transformation systems over PLG.

Choosing the class $\mathcal{N}_{\mathrm{inj}} = \mathcal{M} \cap \mathcal{N}$ of injective $\mathcal{N}$-morphisms, we obtain the following corollary for graph transformation with relabelling and injective matching.

Corollary 2. The category PLG is $\mathcal{M},\mathcal{N}_{\mathrm{inj}}$-adhesive and satisfies the HLR$^+$-properties as well as the $\mathcal{E}'$-$\mathcal{M} \cap \mathcal{N}$ pair factorisation. The Local Church-Rosser Theorem, the Parallelism Theorem, the Concurrency Theorem, and the construction of E-related transformations hold for $\mathcal{M},\mathcal{N}_{\mathrm{inj}}$-adhesive transformation systems over PLG.

Remark 6. $\mathcal{M},\mathcal{N}$-adhesive transformation systems over PLG provide a foundation for the semantics of the graph programming language GP [19, 20]. The graphs on which GP programs operate are totally labelled, and instances of GP's conditional rule schemata are rules with application conditions whose left- and right-hand graphs L and R are also totally labelled. The interface graph K consists of unlabelled nodes and hence enables relabelling of nodes. Moreover, the requirement that the vertical morphisms in double-pushouts must preserve unlabelled nodes guarantees that pushout complements are unique (see [15]).

In comparison with the approach of [15], $\mathcal{M},\mathcal{N}$-adhesive tranformation systems over PLG are more restrictive in that unlabelled nodes in rules must not match labelled nodes in host graphs. However, to allow certain nodes in rules to match nodes with arbitrary labels, one can use rule schemata with label variables instead of unlabelled nodes. As in GP, rule schemata are instantiated to rules with totally labelled left- and right-hand graphs, while unlabelled nodes are solely used for relabelling. Indeed, label variables in left-hand graphs are more versatile than unlabelled nodes because they can be typed in order to match only subsets of labels.

5 Conclusion

Double-pushout graph transformation with relabelling is not covered by $\mathcal{M}$-adhesive transformation systems. Relabelling is natural for computing with graphs, though, and provides a foundation for graph transformation languages such as GP. We have generalised $\mathcal{M}$-adhesive transformation systems to $\mathcal{M},\mathcal{N}$-adhesive transformation systems which do cover graph transformation with relabelling. We have proved the Local Church-Rosser Theorem, the Parallelism Theorem and the Concurrency Theorem for $\mathcal{M},\mathcal{N}$-adhesive transformation systems with application conditions, and hence these results hold for graph transformation with relabelling. An overview over the used $\mathcal{M},\mathcal{N}$-adhesive, HLR, and HLR$^+$ properties is given below. HLR, Shift, Left, LCR, Par, Con, and Const stand for HLR-properties, Shift over morphisms, Shift over rules, Local Church Rosser, Parallelism, Concurrency, and Construction of E-related tranformations.

$\mathcal{M},\mathcal{N}$-adhesive properties	HLR	Shift	Left	LCR	Par	Con	Const
$\mathcal{M}$ closed under composition						x	
$\mathcal{N}$ closed under composition						x	x
$\mathcal{N}$ closed under $\mathcal{M}$-decomposition			x	x			
$\mathcal{C}$ has $\mathcal{M},\mathcal{N}$-POs			x	x	x	x	–
$\mathcal{C}$ has $\mathcal{M}$-PBs			x	x	x	x	x
$\mathcal{M}$ stable under POs & PBs			x	x	x	x	–
$\mathcal{N}$ stable under POs & PBs			x	x	x	x	x
$\mathcal{M},\mathcal{N}$-PO are $\mathcal{M},\mathcal{N}$-VK squares	x						
HLR-properties							
$\mathcal{M},\mathcal{N}$-POs are PBs			x				
$\mathcal{M},\mathcal{N}$-PO-PB decomposition			x	x			x
Cube $\mathcal{M},\mathcal{N}$-PO-PB decomp						x	x
Uniqueness of PO complements						x	
HLR$^+$-properties							
$\mathcal{C}$ has binary coproducts					x		
$\mathcal{C}$ has $\mathcal{E}$-$\mathcal{N}$ factorization					x		
$\mathcal{C}$ has $\mathcal{E}'$-$\mathcal{M}$ factorization		x					x

We hope to establish the Amalgamation Theorem, the Embedding Theorem and the Local Confluence Theorem in our new framework, too. These results have recently been proved for $\mathcal{M}$-adhesive transformation systems with application conditions [6, 7].

In future work, we expect to be able to show that the category of term graphs is $\mathcal{M}, \mathcal{N}$-adhesive. This category is known to be not $\mathcal{M}$-adhesive, too, but has been shown to be quasi-adhesive [2]. Indeed the categories of term graphs and partially labelled graphs are similar in that PLG can also be shown to be quasi-adhesive. In PLG, the regular monomorphisms are precisely the undefinedness preserving injective morphisms.

An extension of $\mathcal{M}, \mathcal{N}$-adhesive transformation systems with rules that have a non-monomorphic right-hand morphism, allowing to merge items, may be possible. In the context of graph transformation with relabelling, the approach of [15] already includes such rules. Independently, in [1] a class of categories is identified for which the local Church-Rosser property holds for certain classes of rules with non-monomorphic right-hand morphisms.

Finally, the $\mathcal{W}$-adhesive transformation systems introduced in [11] provide a general framework for attributed objects. They allow undefined attributes in the interface of a rule to change attributes, which is similar to relabelling. But the precise relationship to $\mathcal{M}, \mathcal{N}$-adhesive transformation systems remains to be worked out.

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